A probabilistic Kleene Theorem

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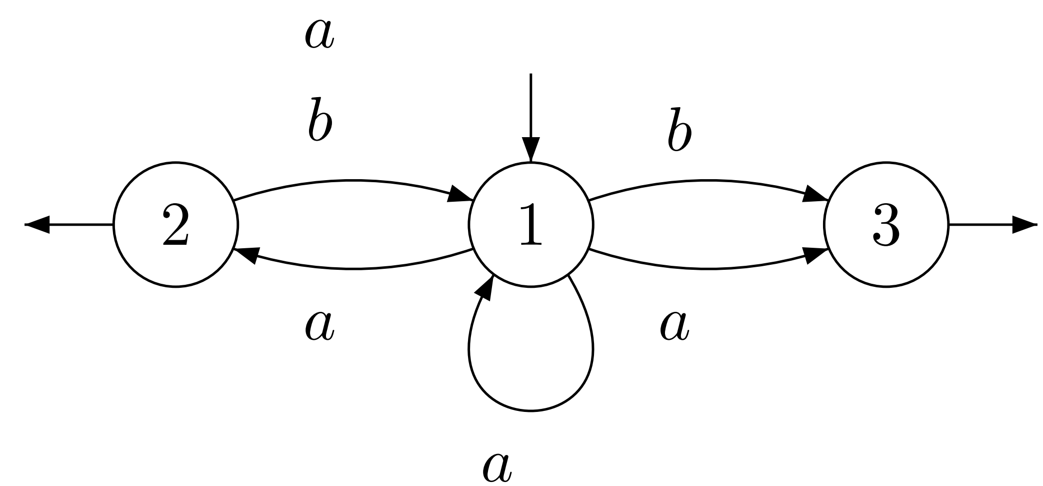
Part of works published at ATVA'12 with Benedikt Bollig, Paul Gastin and Marc Zeitoun

Kleene's Theorem

Finite State Automata



same expressivity

Regular Expressions

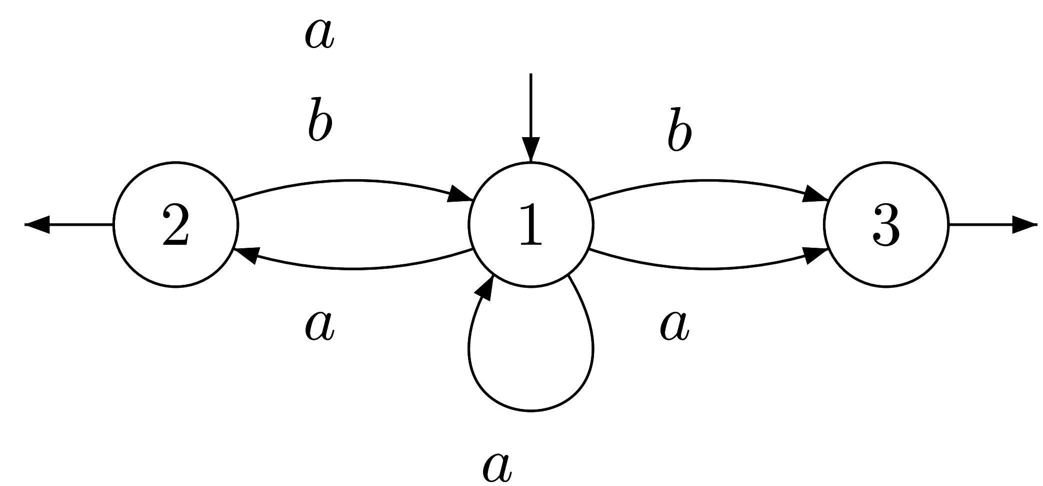
E ::= a | E+E | E · E | E*

Motivations

- Theoretically: relate denotational and computational models
- Practically: easier to write specifications using regular expressions vs. easier to check properties (emptiness, inclusion...) with automata
- Goal: translate expressions to automata, as efficiently as possible

Kleene's Theorem

Finite State Automata



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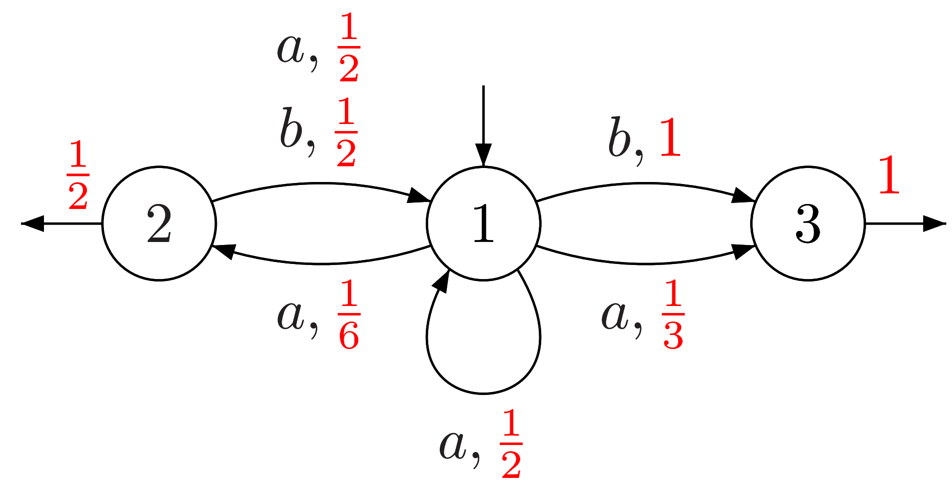
$$E ::= \quad a \mid E+E \mid E \cdot E \mid E^*$$

Kleene's Theorem

Weighted Finite State Automata



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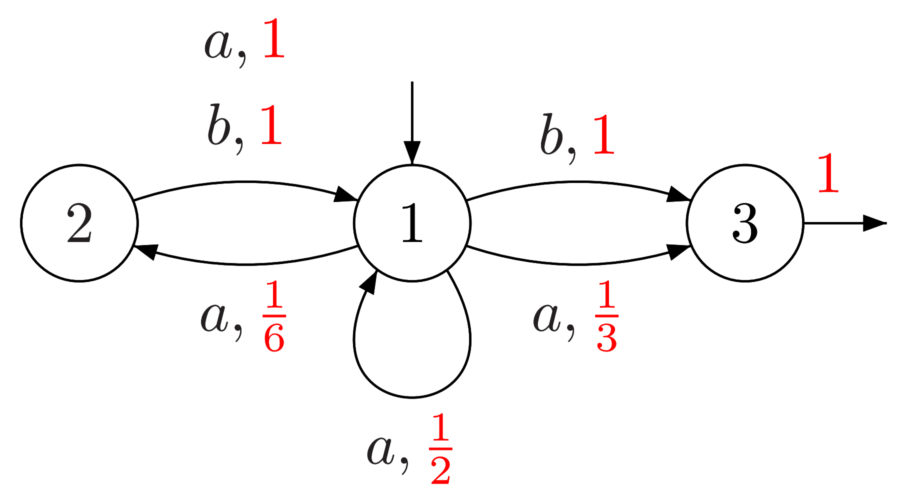

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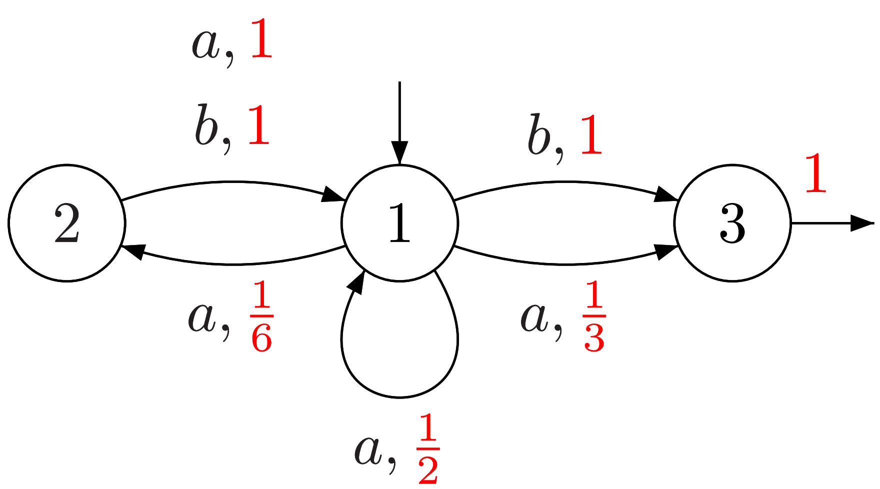

$\Sigma^* \rightarrow \mathbb{R}$

same expressivity

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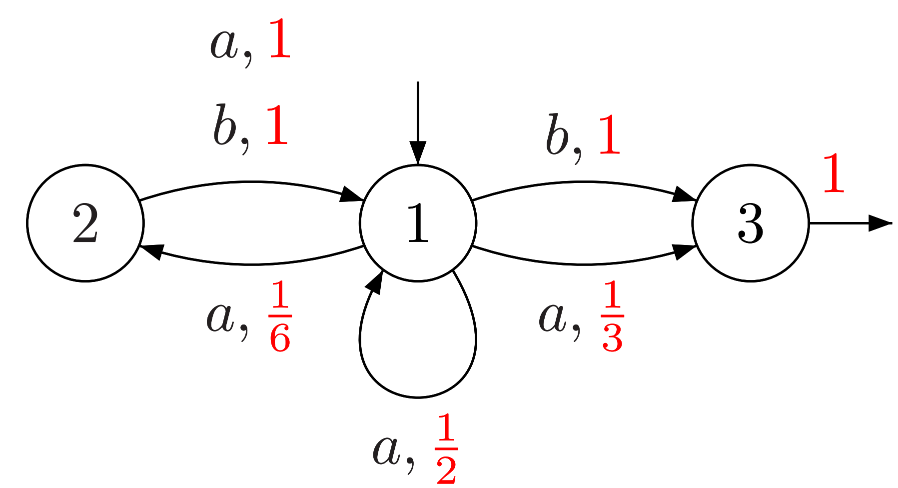

$\Sigma^* \rightarrow \mathbb{R}$

same expressivity

a a b

two runs:
1→2→1→3 of weight 1/6x1x1x1=1/6
and 1→1→1→3 of weight 1/2x1/2x1x1=1/4

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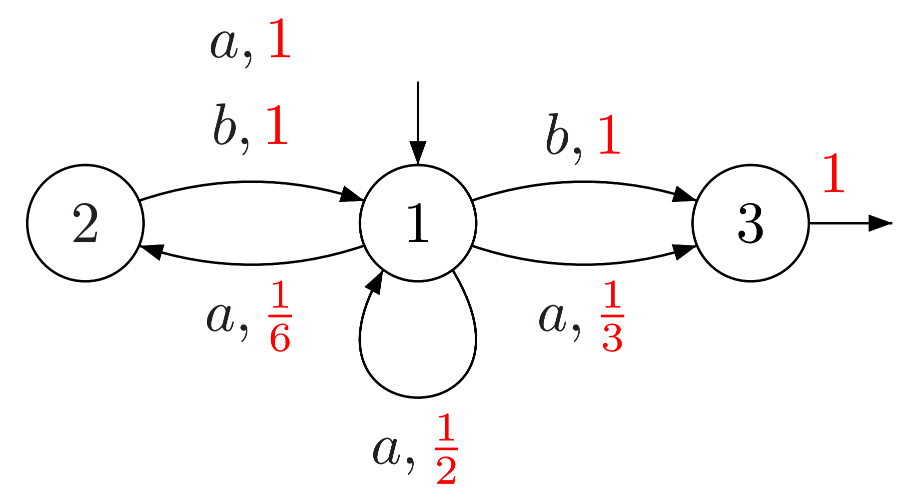
Weighted Finite State Automata



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hence, *a a b* recognized with weight 1/6+1/4=5/12

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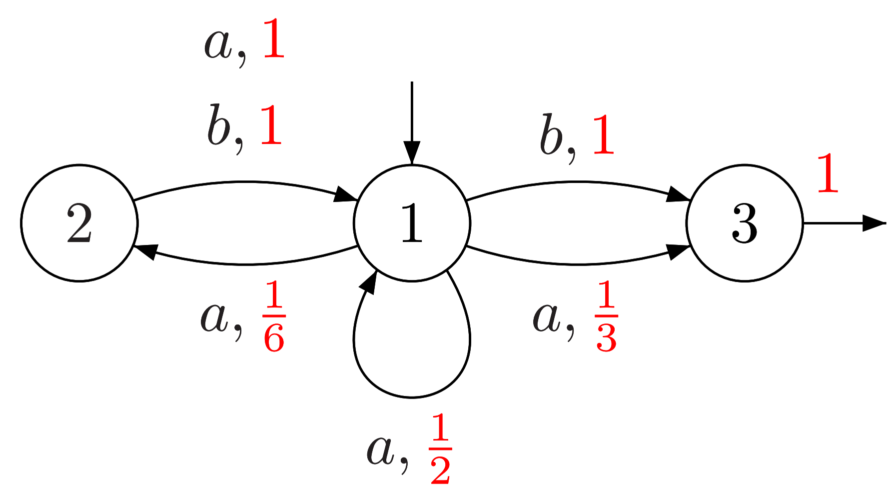
Weighted Finite State Automata



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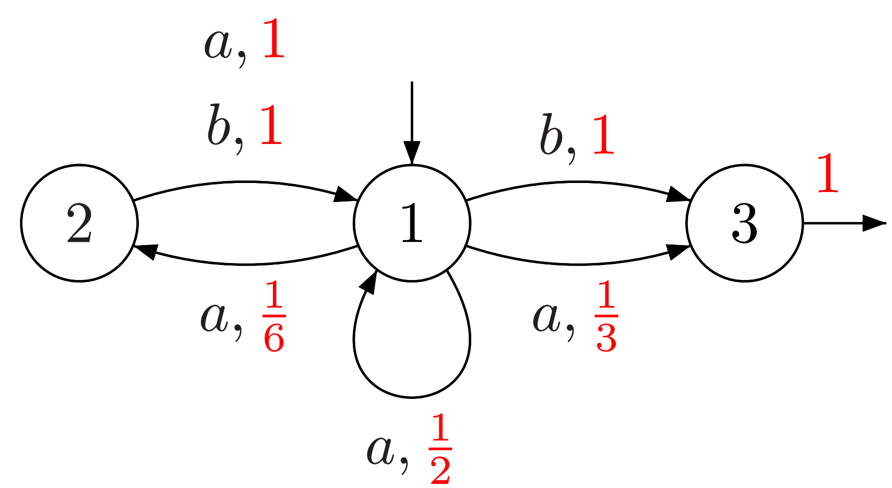
Weighted Regular Expressions

E ::= *p* | *a* | E+E | E · E | E* ← E proper

Schützenberger Kleene's Theorem

Weighted Finite State Automata



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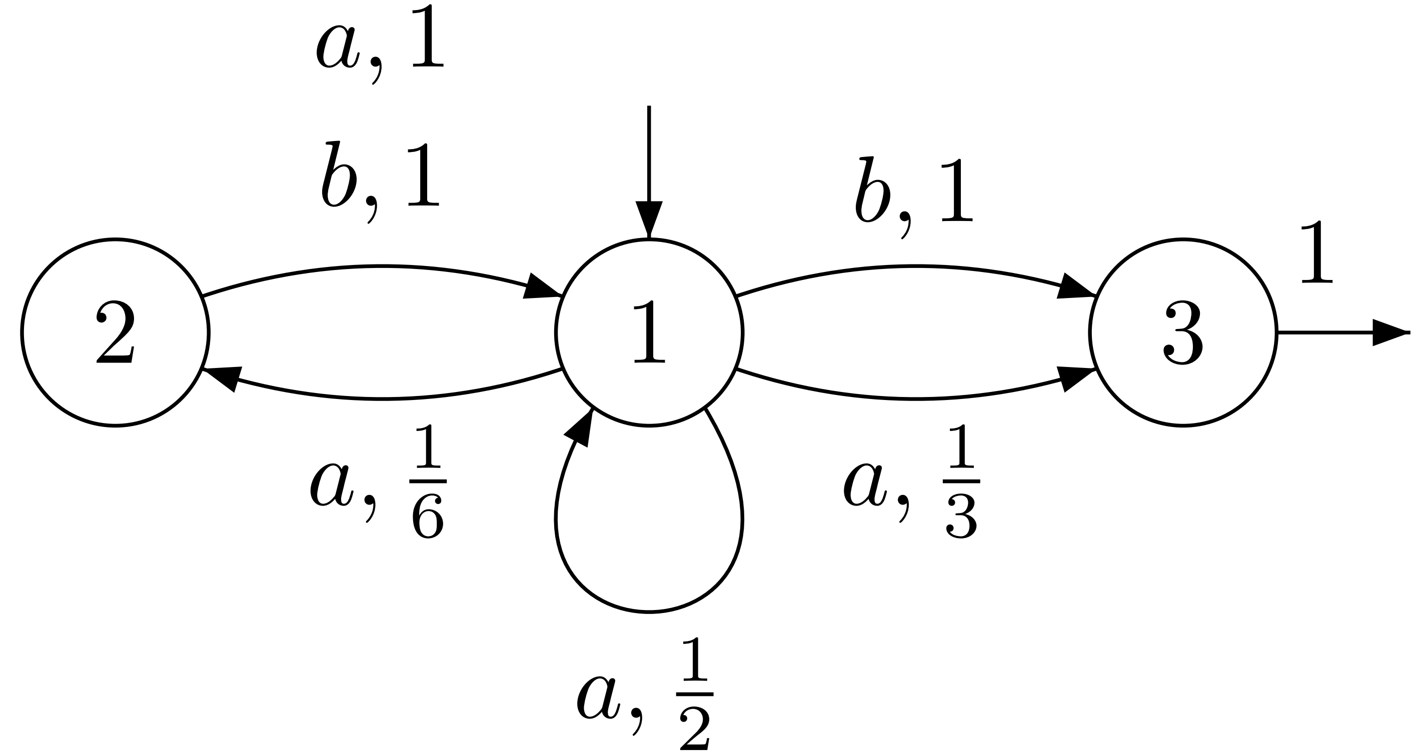
[1] S. Kleene (1956). Representation of events in nerve nets and finite automata.

[2] M.-P. Schützenberger (1961). On the Definition of a Family of Automata. Information and Control.

For an overview about Weighted Automata, see, e.g., Handbook of Weighted Automata. Editors: Manfred Droste, Werner Kuich, and Heiko Vogler. EATCS Monographs in Theoretical Computer Science. Springer, 2009.

Probabilistic case?



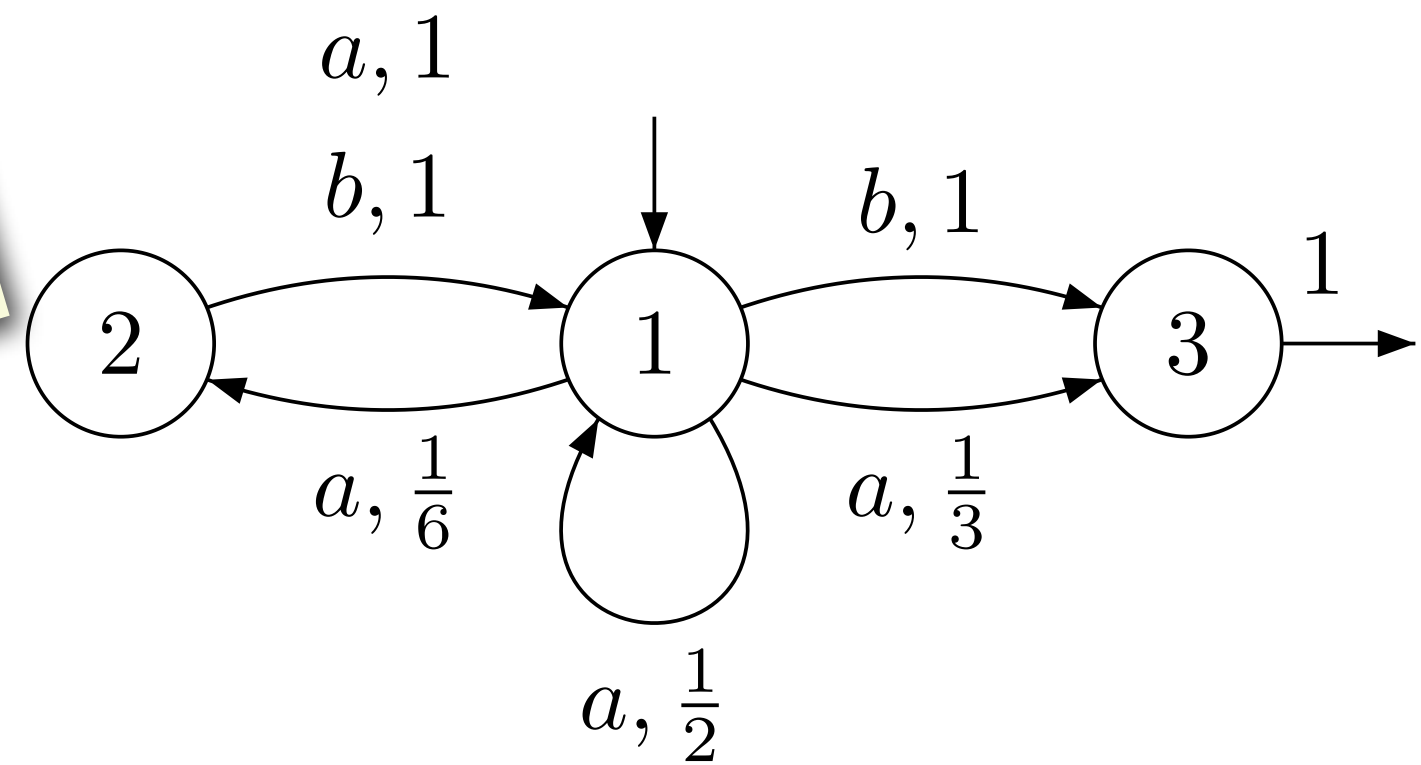
$$\mathcal{A} = (Q, \iota, Acc, \mathbb{P})$$

$$\mathbb{P} : Q \times \Sigma \times Q \to [0, 1]$$

$$Acc(q) + \sum_{q' \in Q} \mathbb{P}(q, a, q') \leq 1 \text{ for all } (q, a) \in Q \times A$$

Probabilistic case?

Reactive Probabilistic Finite Automata



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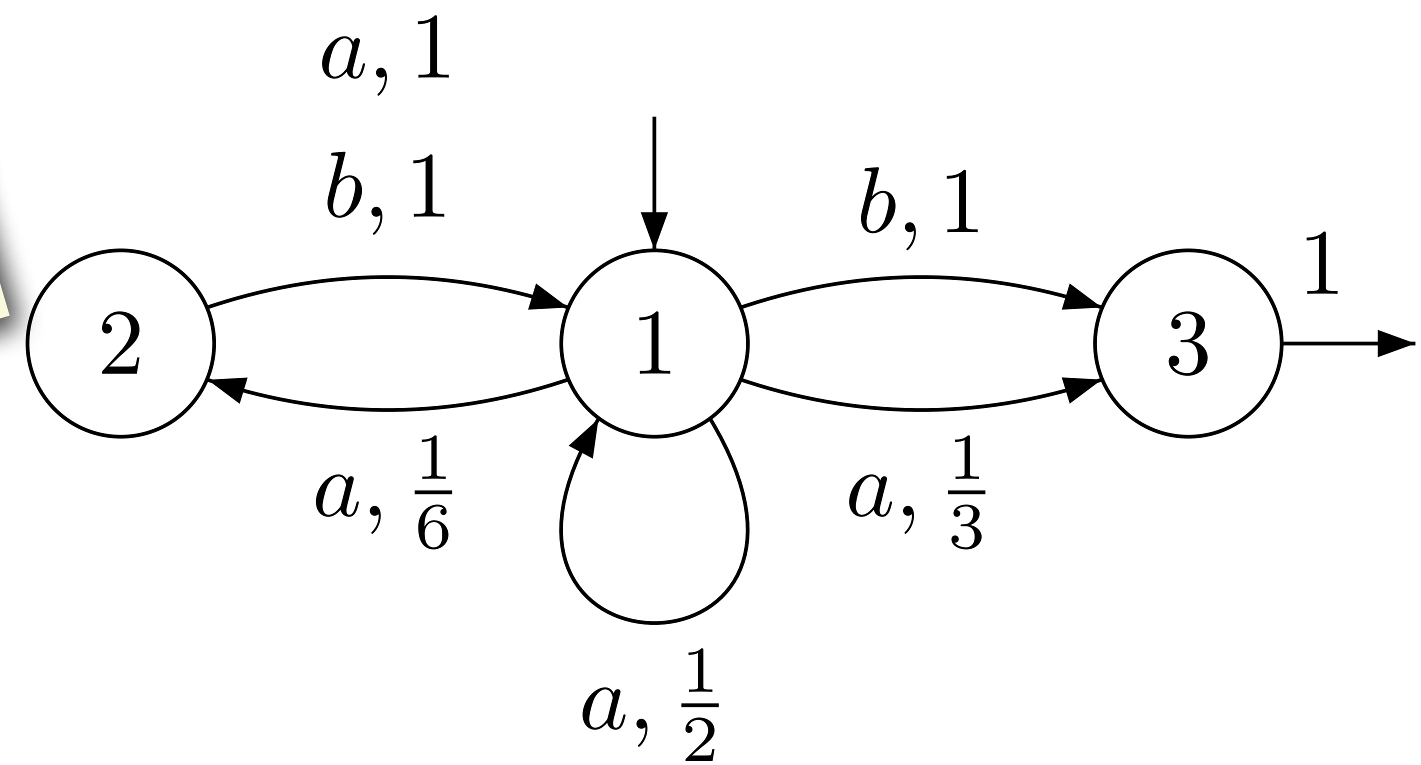


$$\mathcal{A} = (Q, \iota, Acc, \mathbb{P})$$

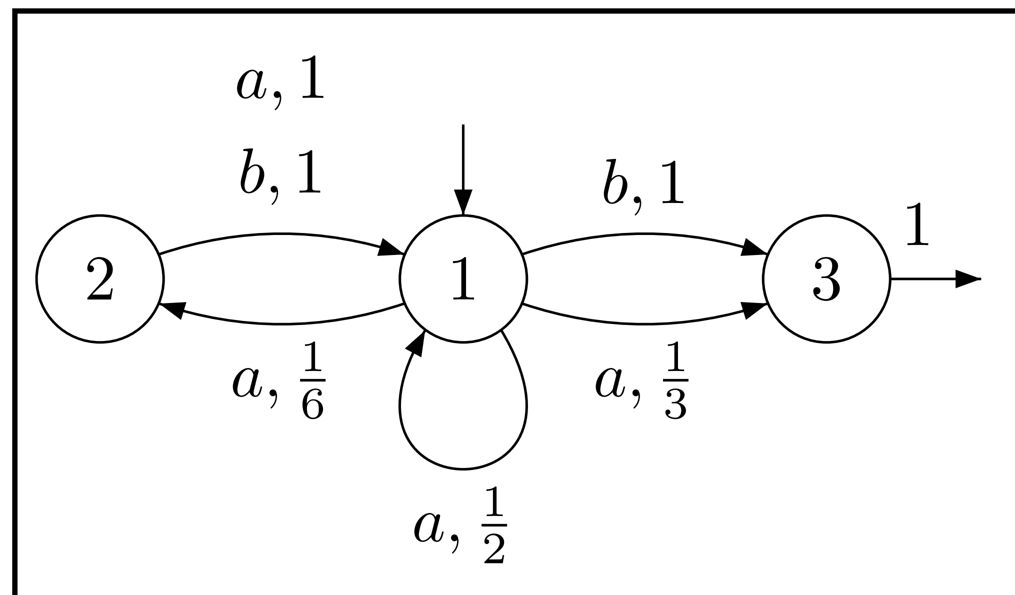
Applying Schützenberger's Theorem over these special Weighted Automata, we obtain regular expressions (proper)

ll $(q, a) \in Q \times A$

$q \in Q$

$$(\tfrac{1}{6}a(a+b) + \tfrac{1}{2}a)^*(\tfrac{1}{3}a + b)$$

What kind of expressions?



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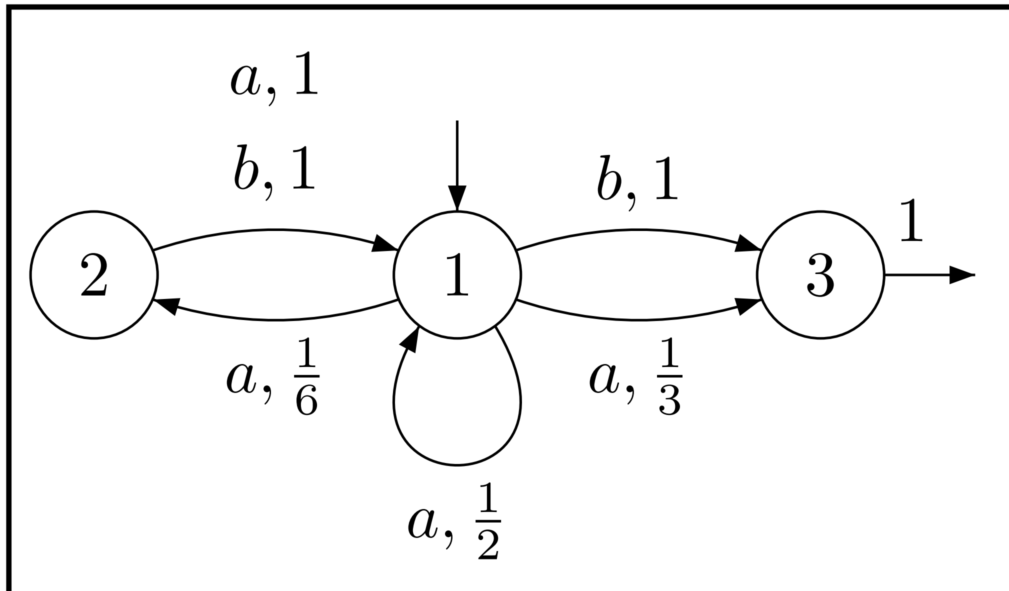

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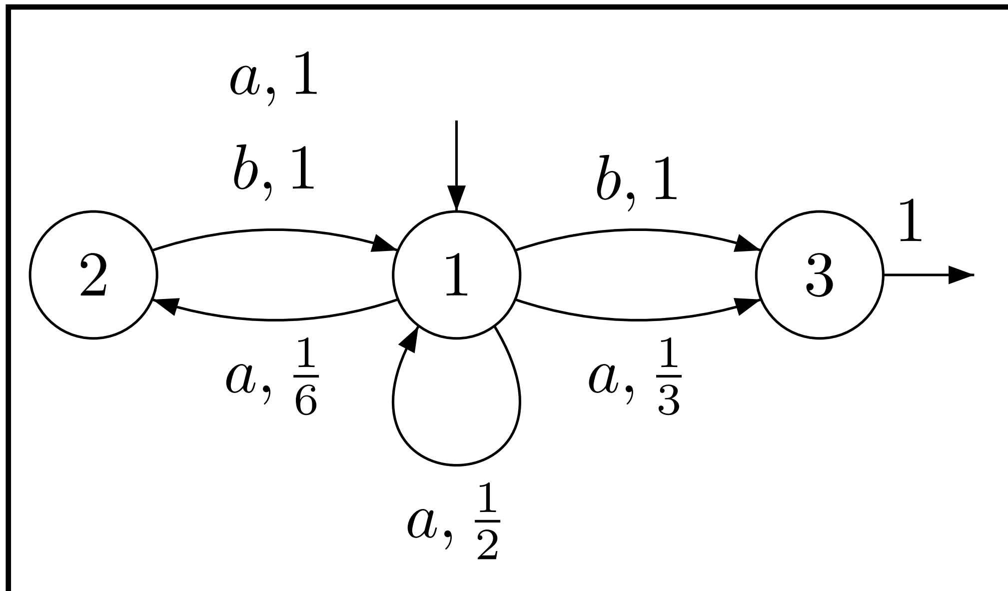

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$$(\tfrac{1}{6}a(a+b)+\tfrac{1}{2}a)^*(\tfrac{1}{3}a+b)$$ ✔

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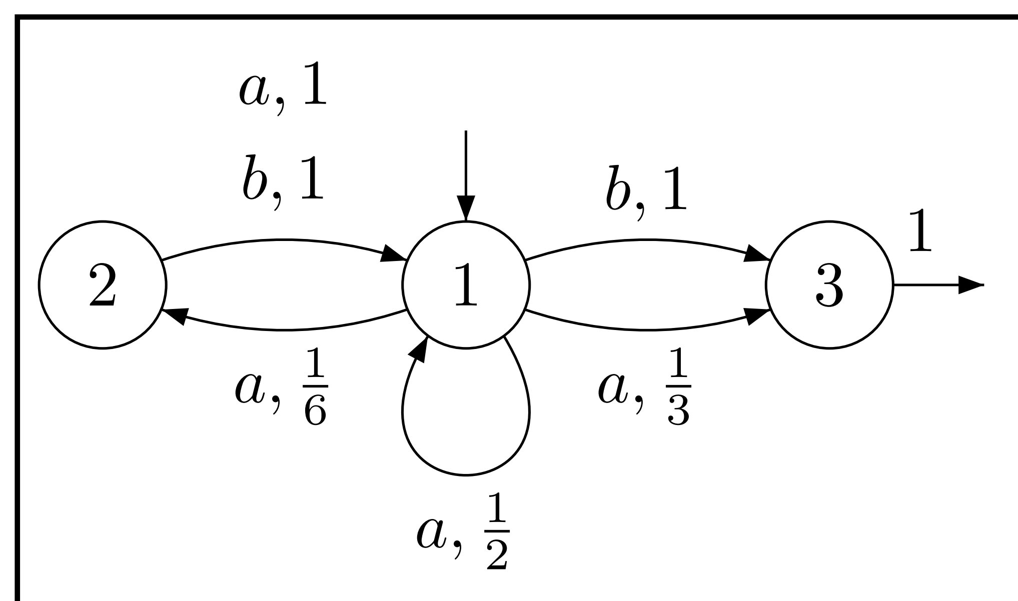

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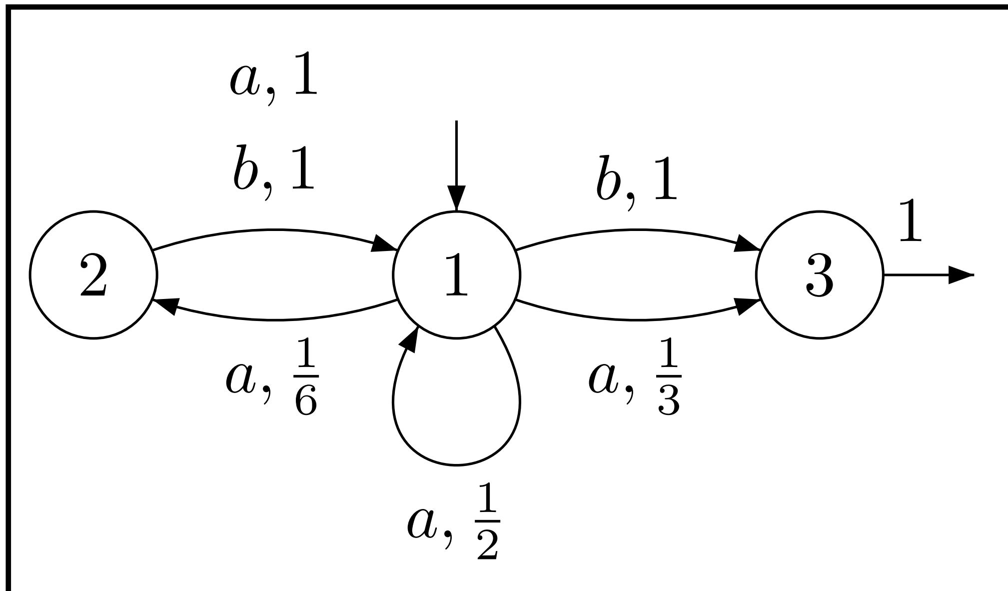

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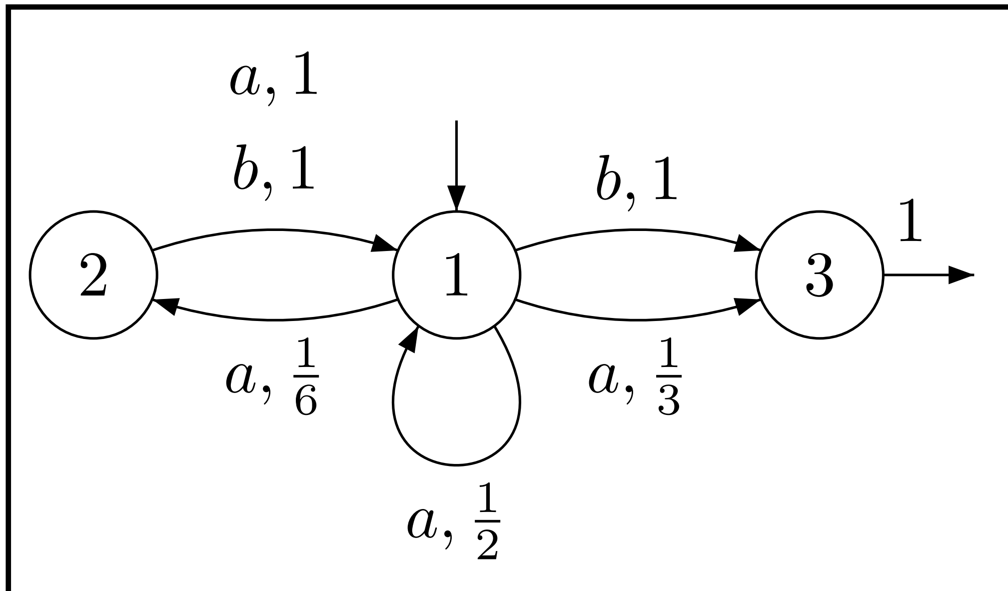

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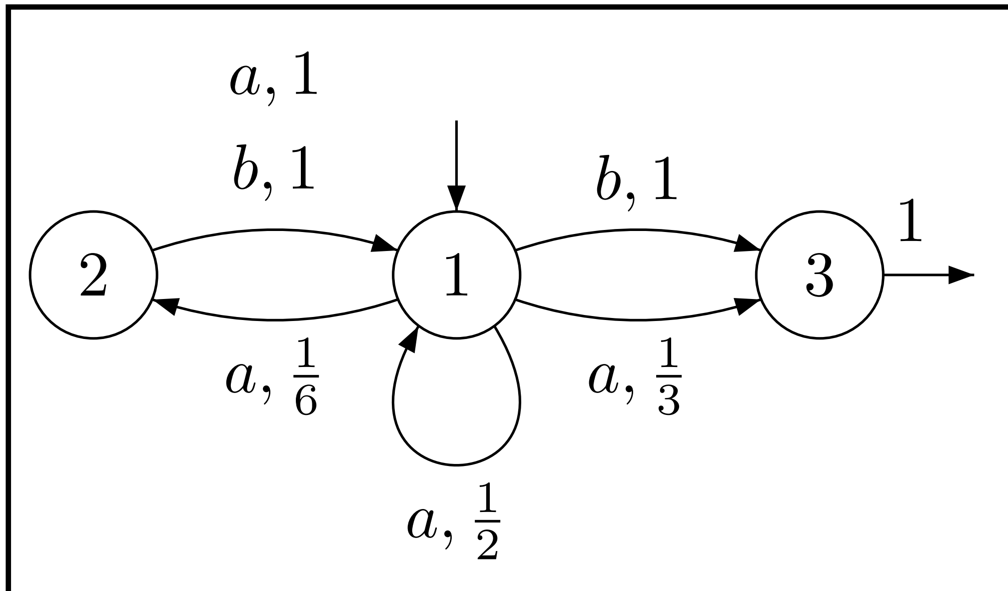

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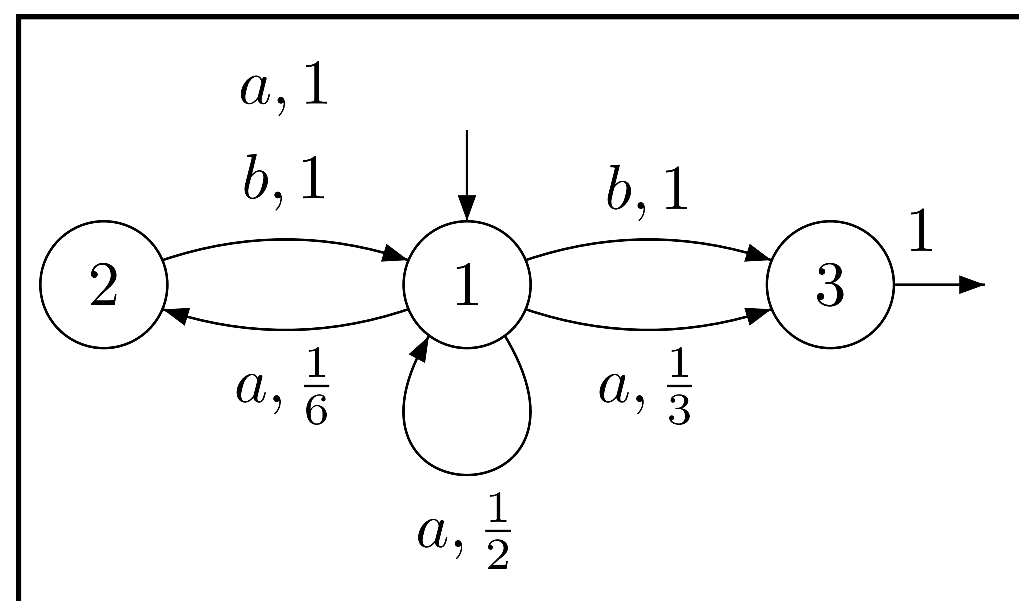
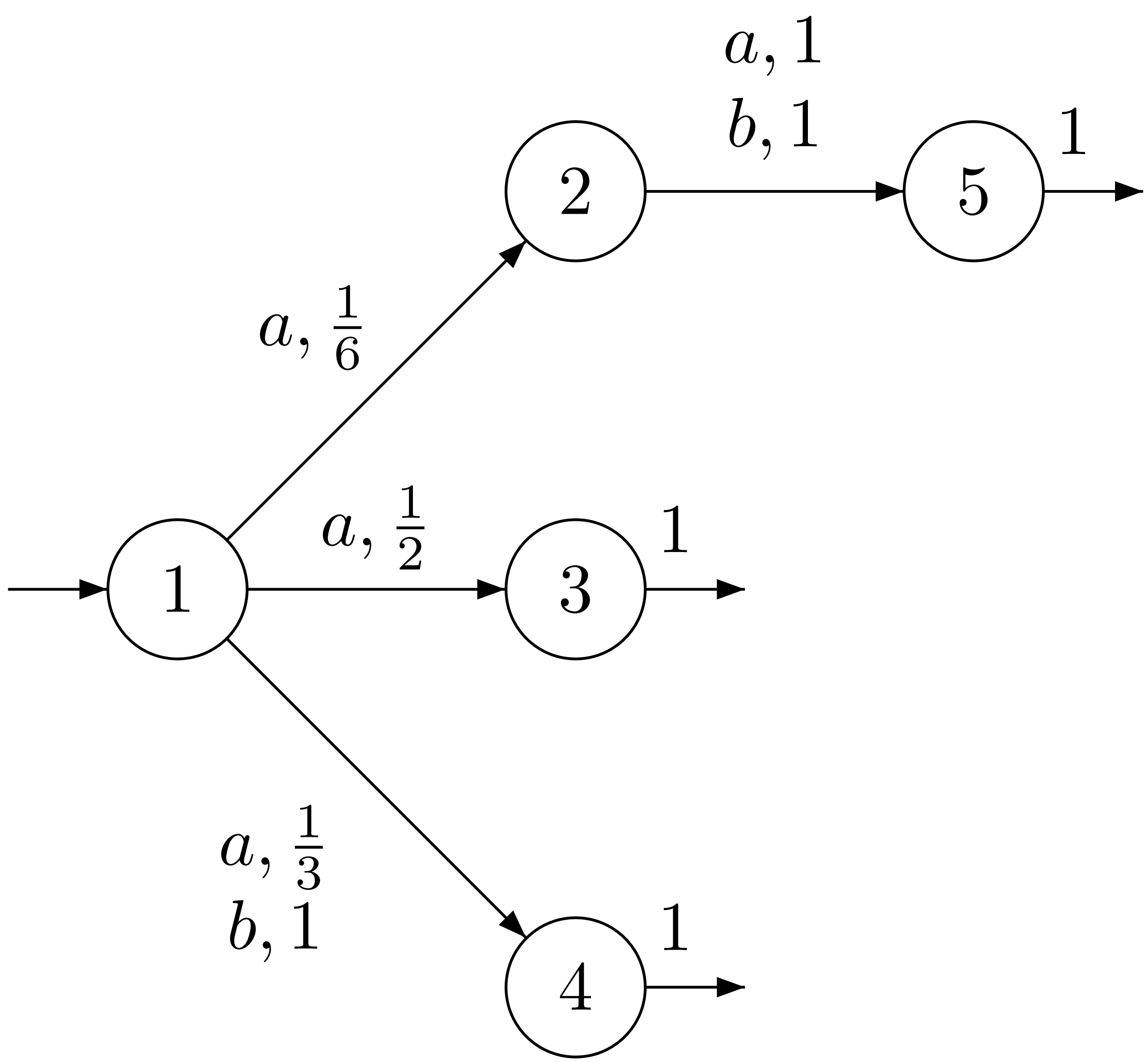
Searching for a natural fragment of weighted regular expressions representing probabilistic behaviors



Constructing Probabilistic Expressions

How to iterate?



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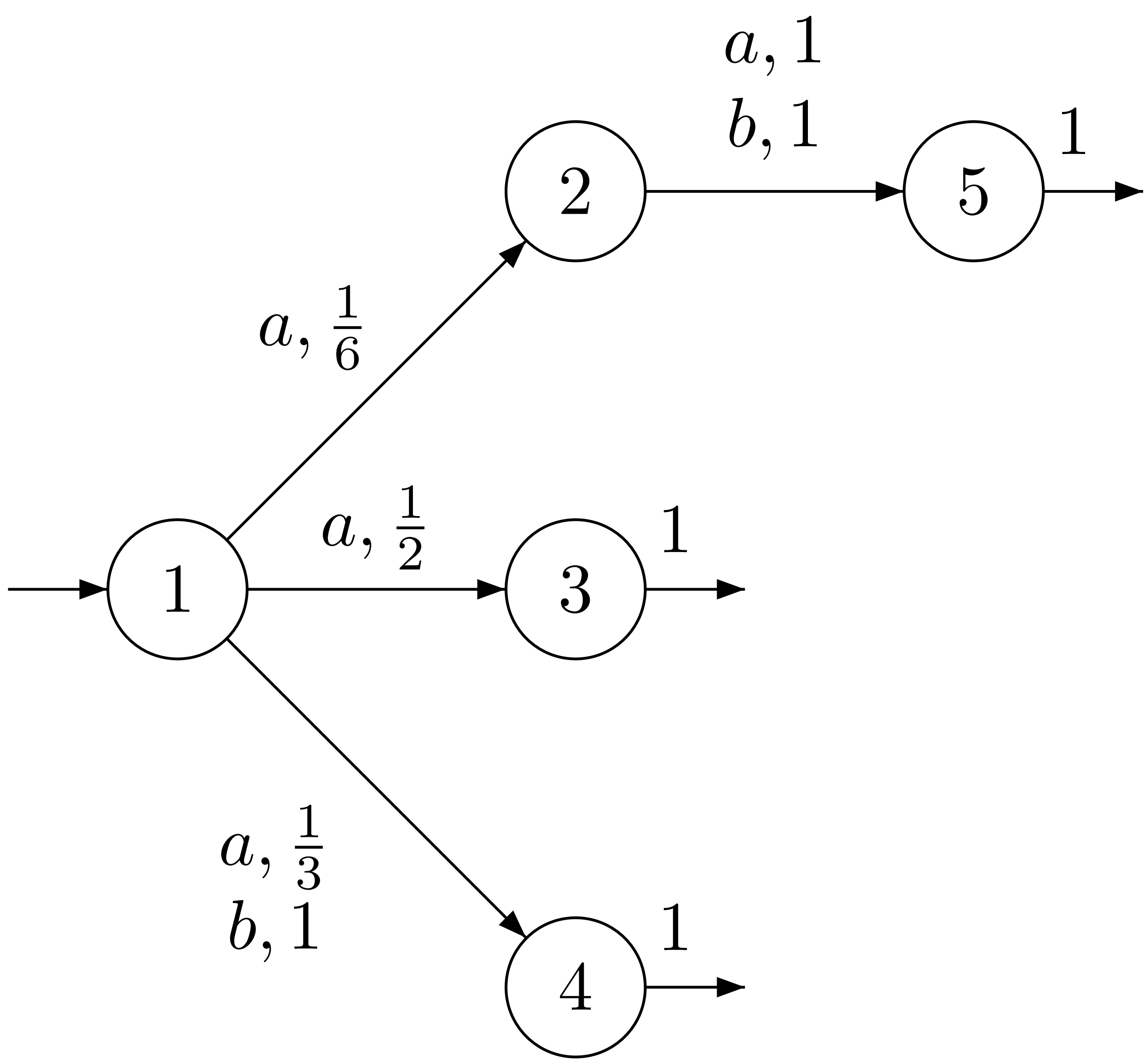

$$\frac{1}{6}a(a+b) + \frac{1}{2}a + (\frac{1}{3}a + b)$$

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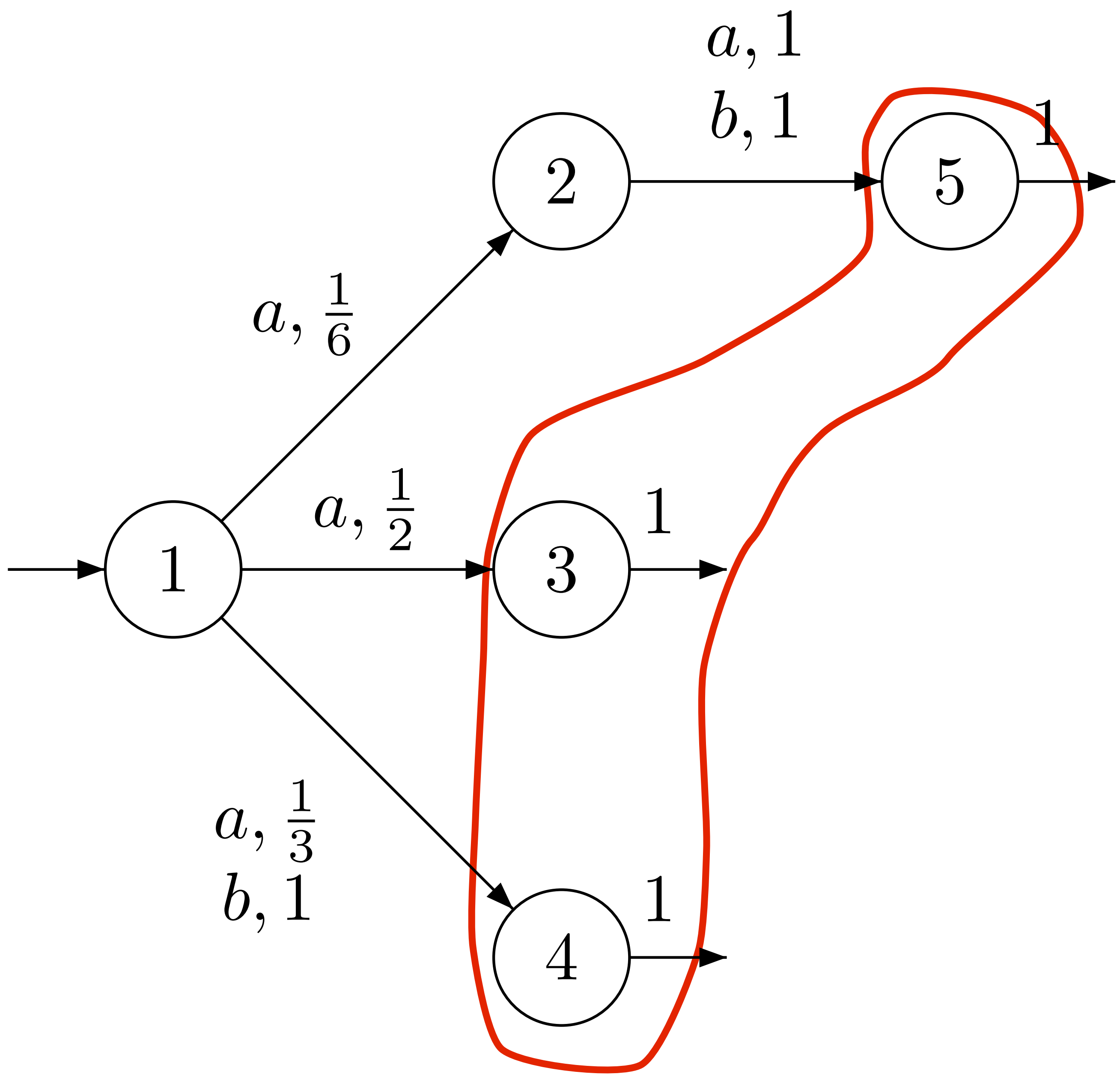

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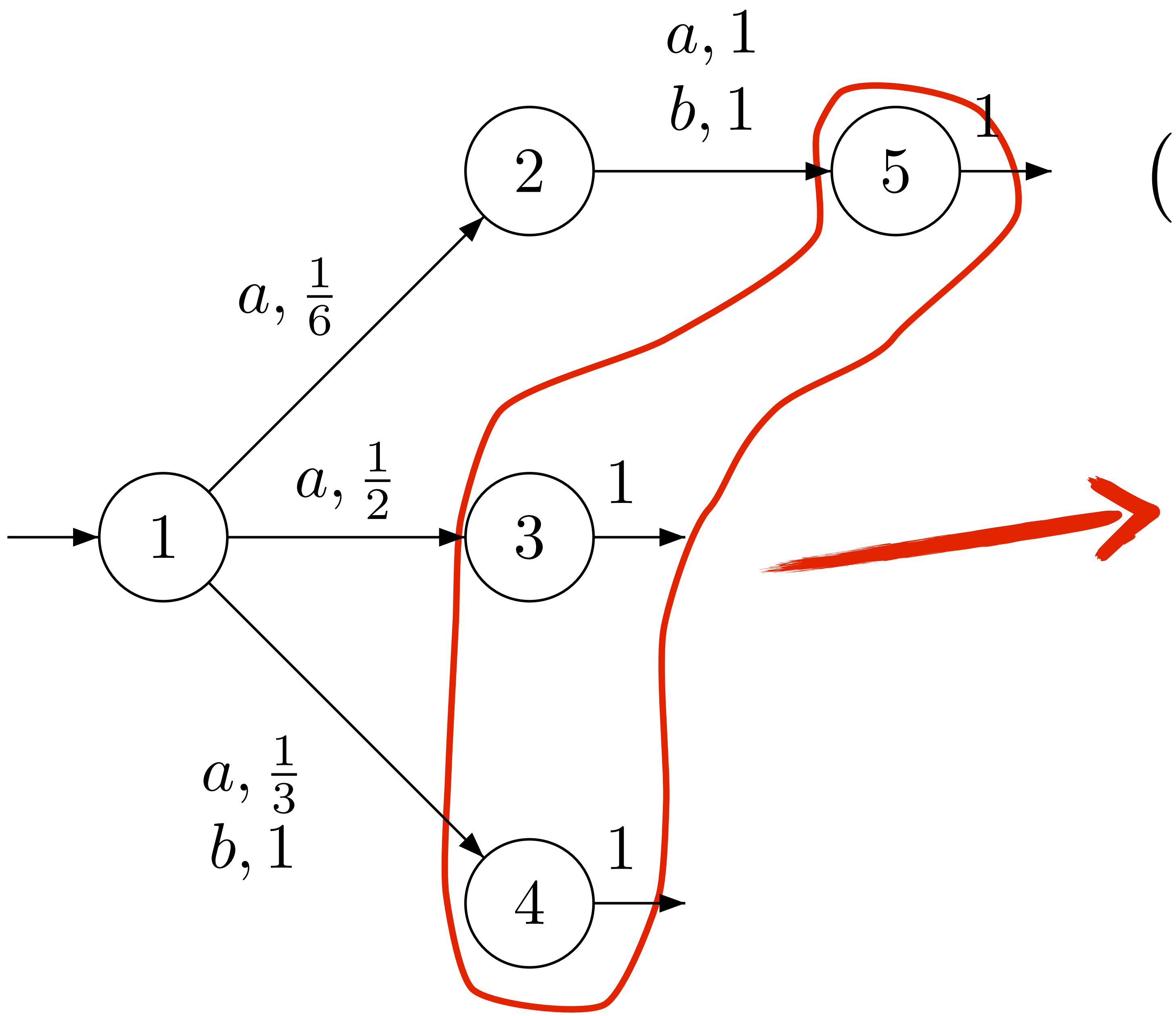

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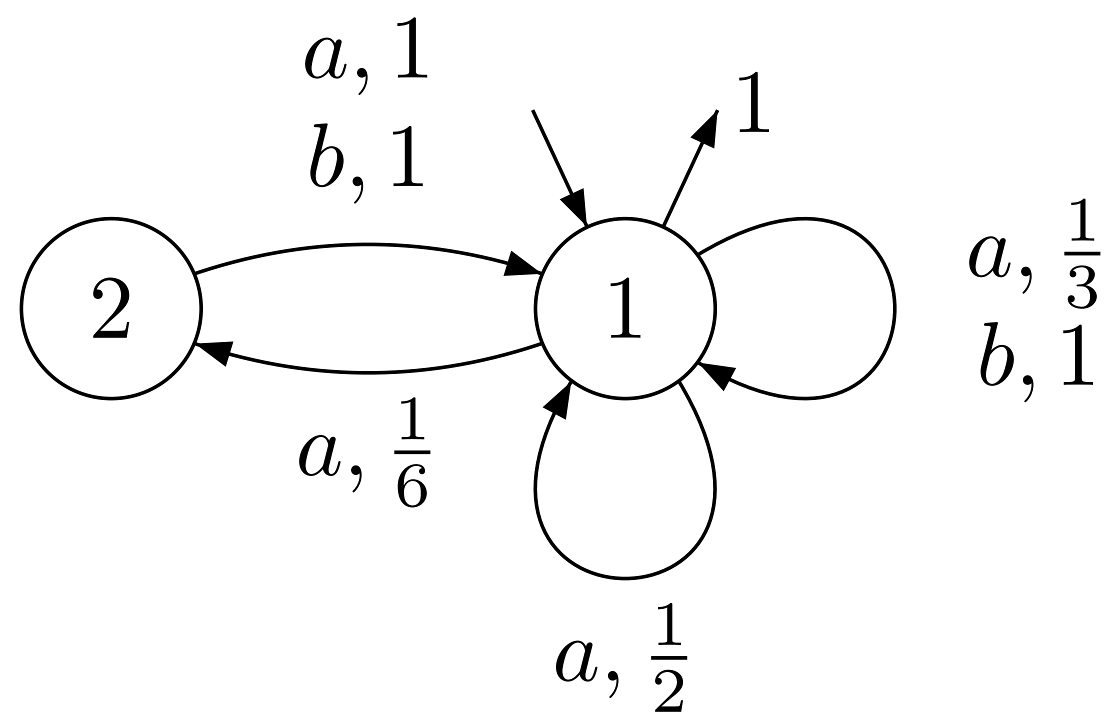
$$\left(\frac{1}{6}a(a+b) + \frac{1}{2}a + (\frac{1}{3}a + b)\right)^*$$



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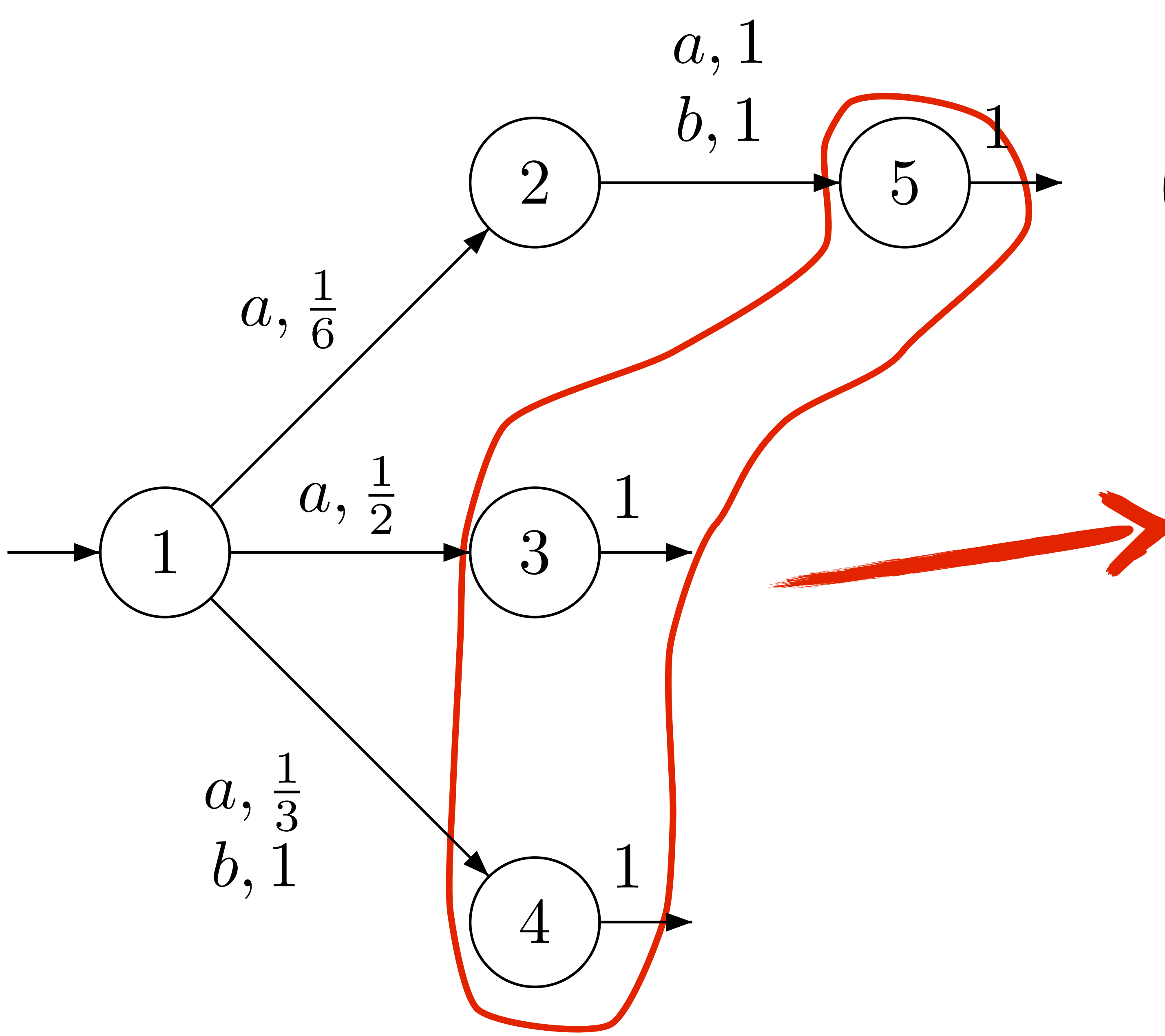

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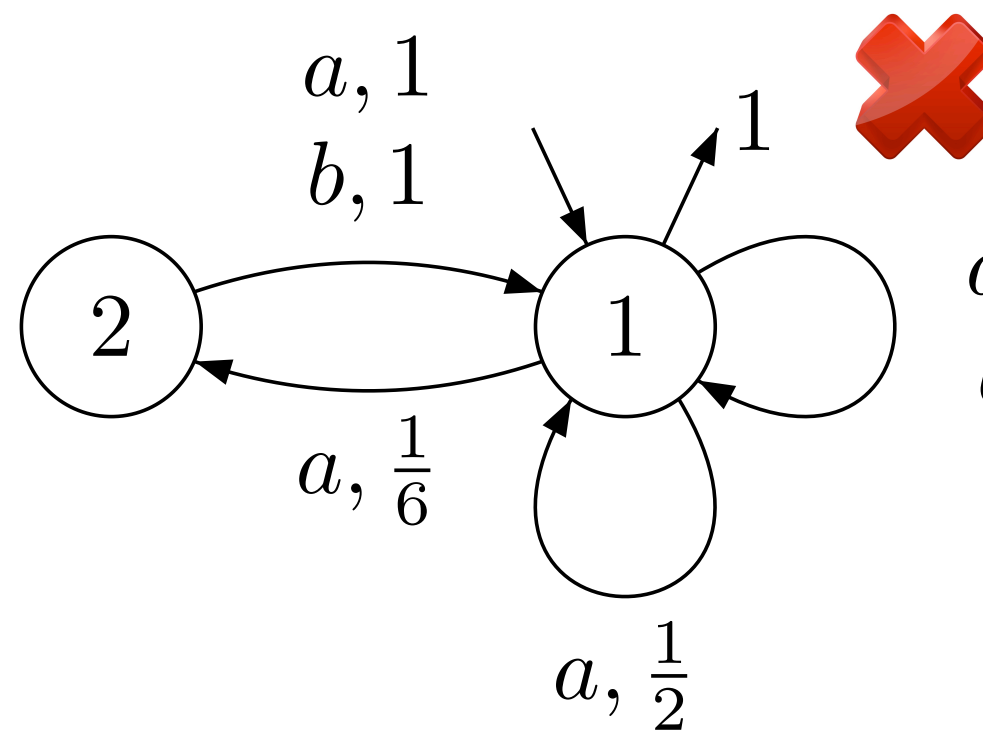

Not a valid Probabilistic Automaton anymore (acceptance condition not fulfilled)

Constructing Probabilistic Expressions

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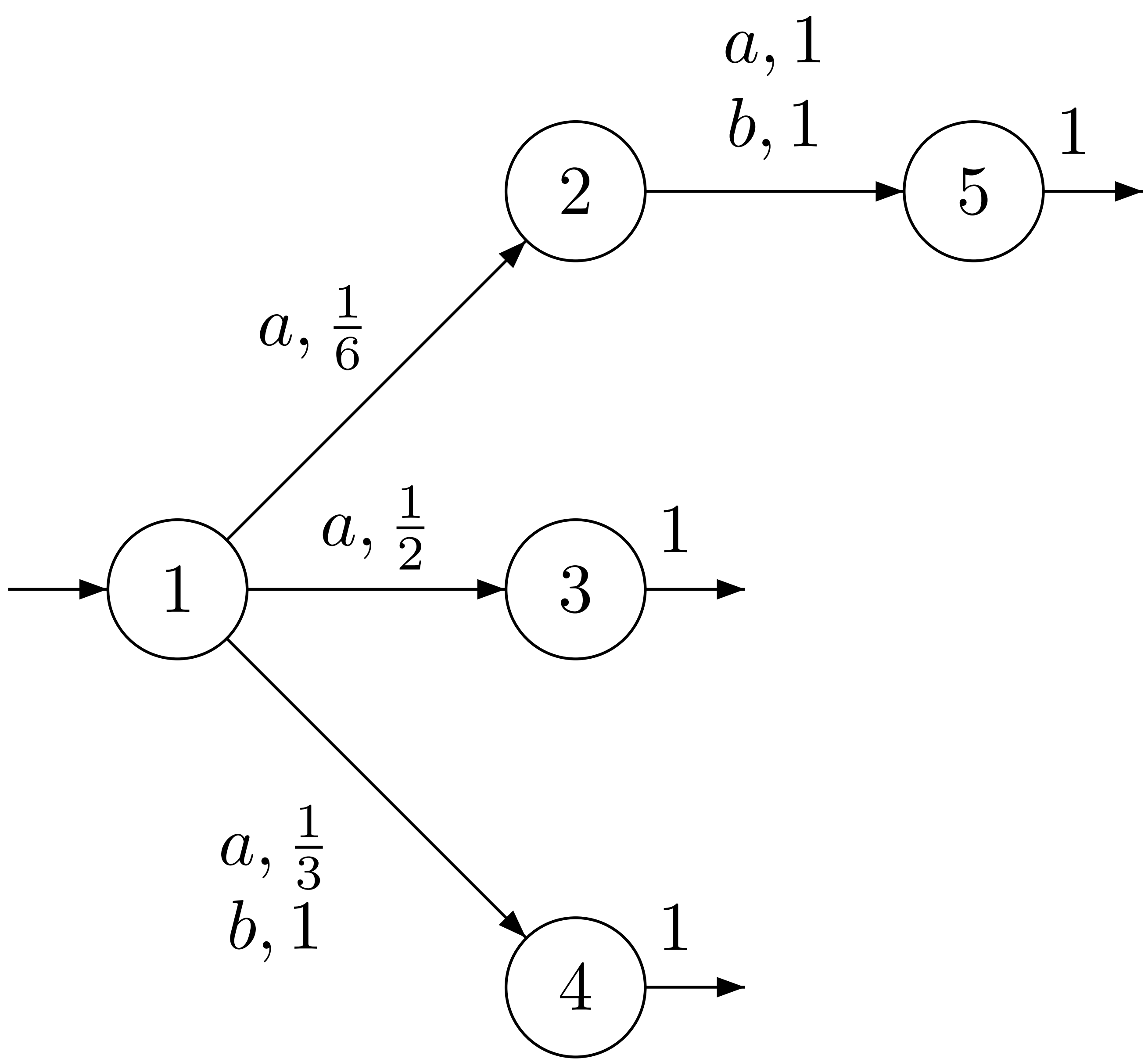

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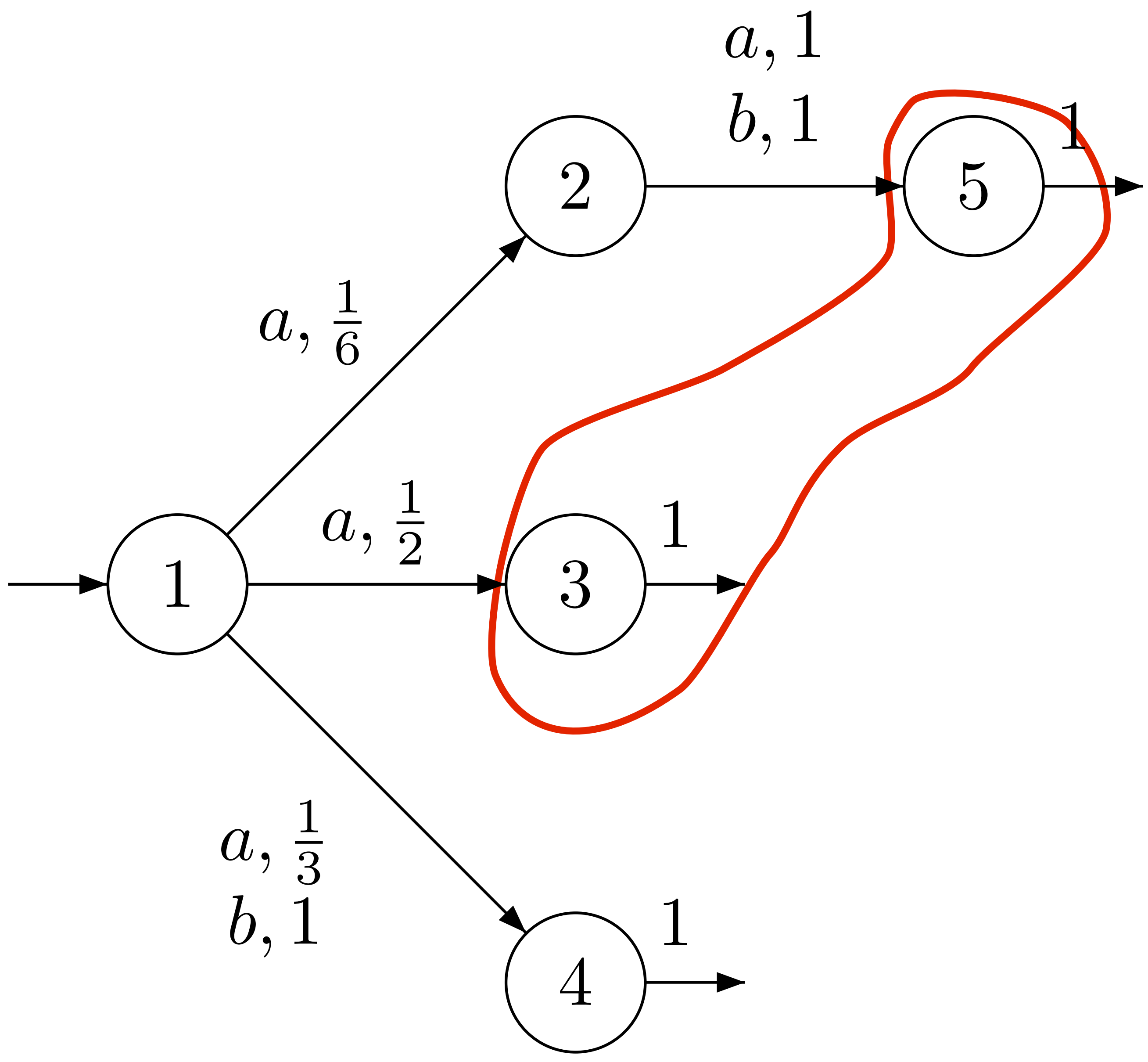

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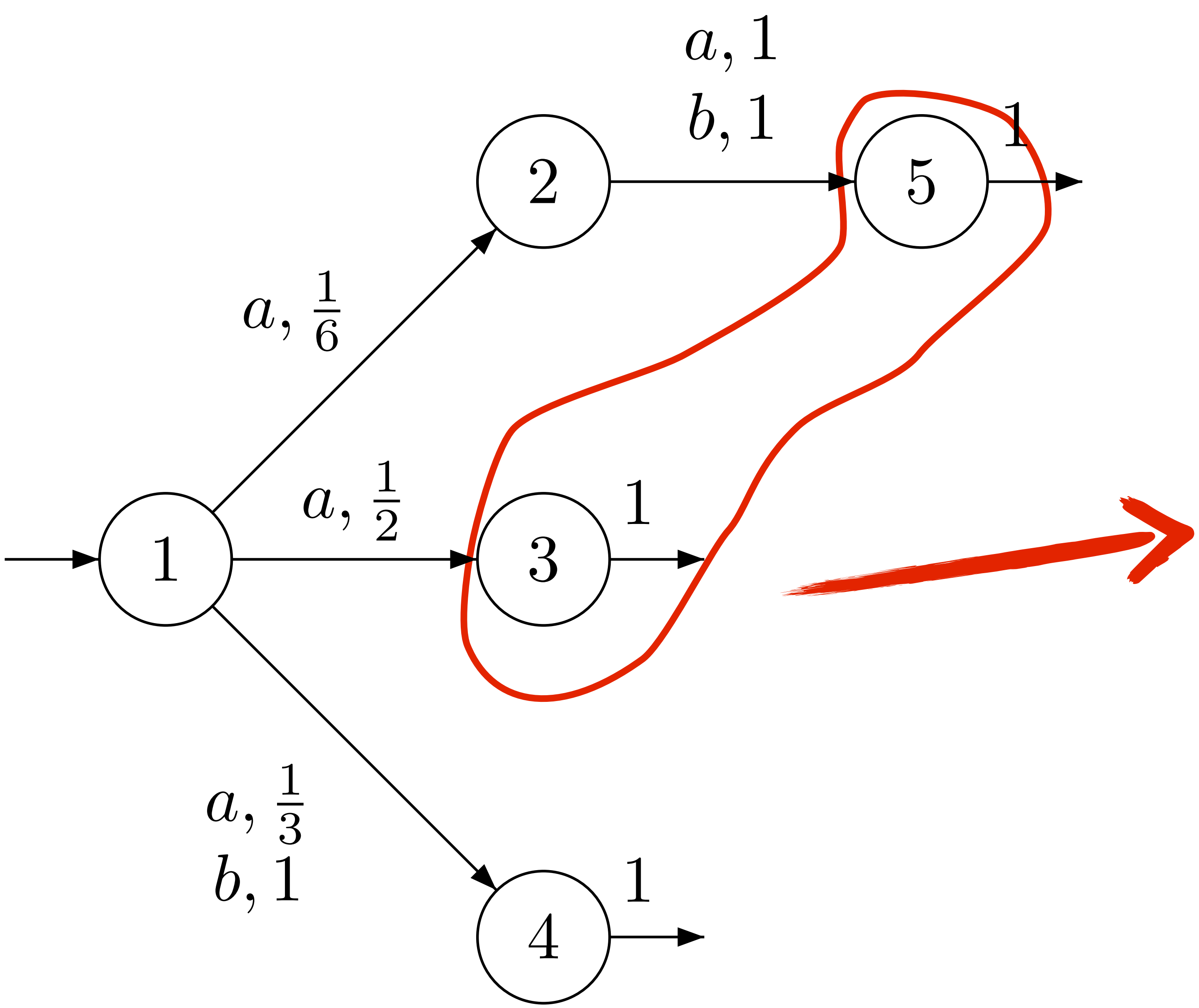

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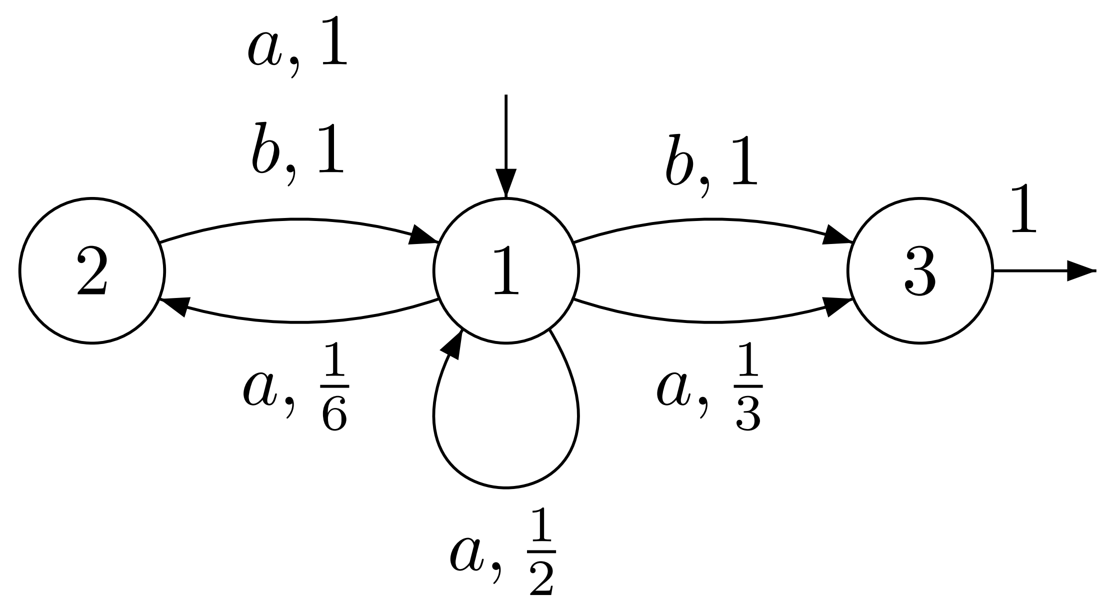

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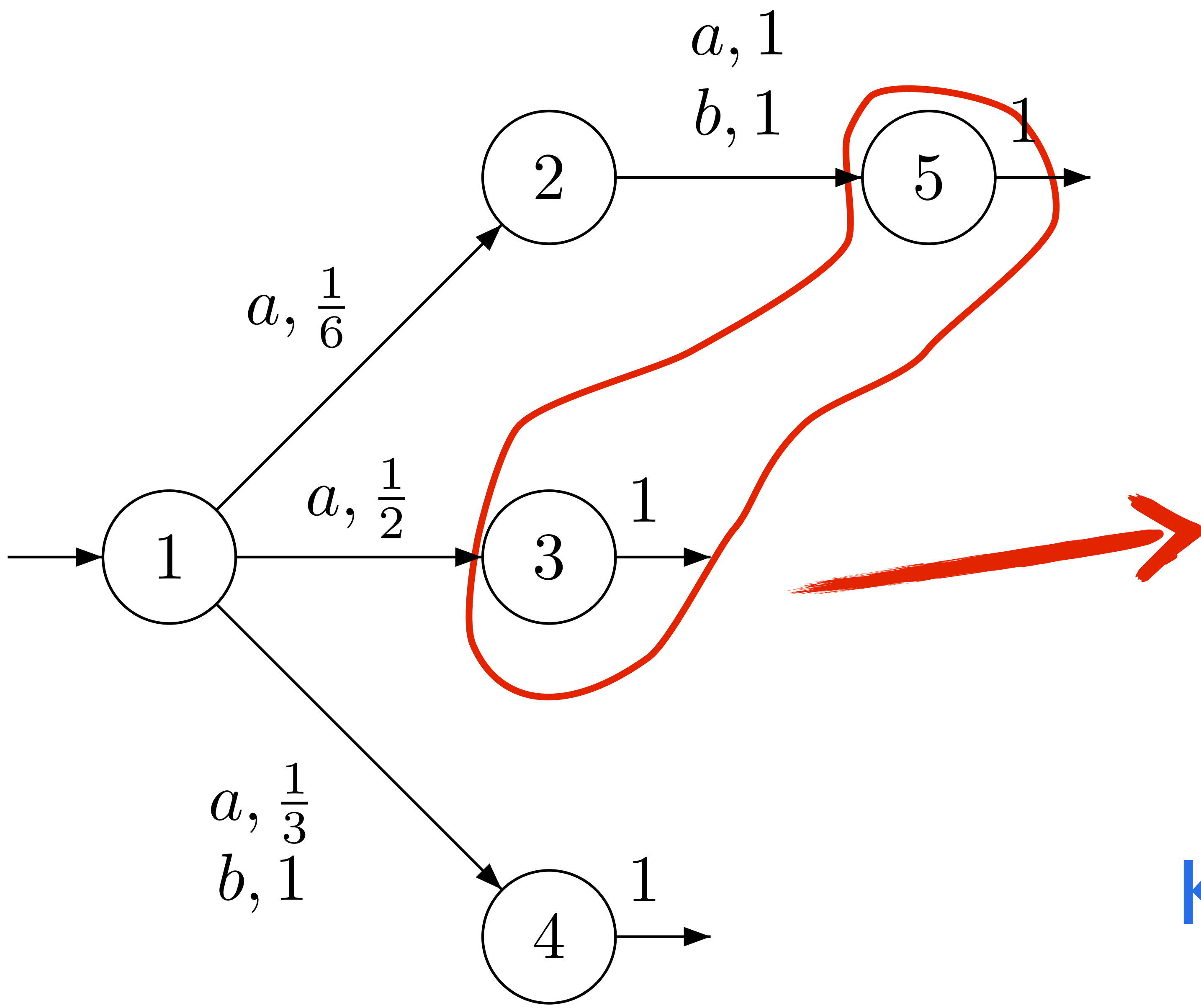

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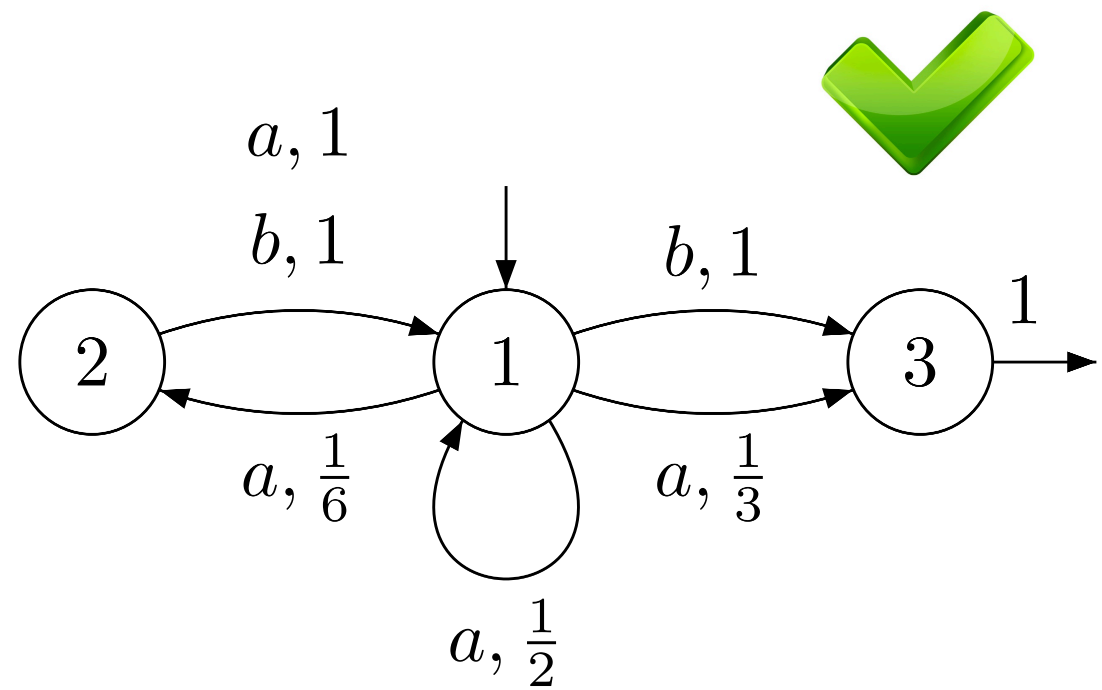

$$\tfrac{1}{6}a(a+b) + \tfrac{1}{2}a + \left(\tfrac{1}{3}a + b\right)$$

Keep some *branch* for termination of the Probabilistic Automaton

Probabilistic Expressions

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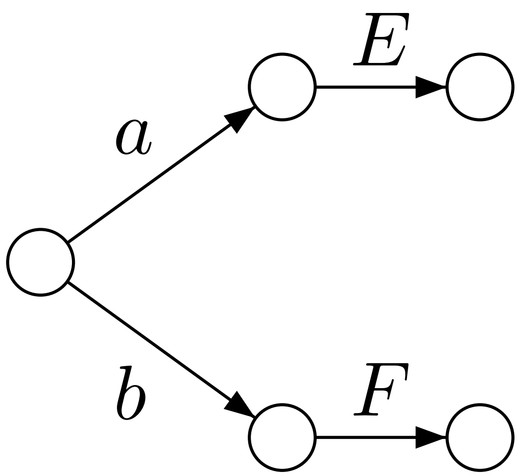

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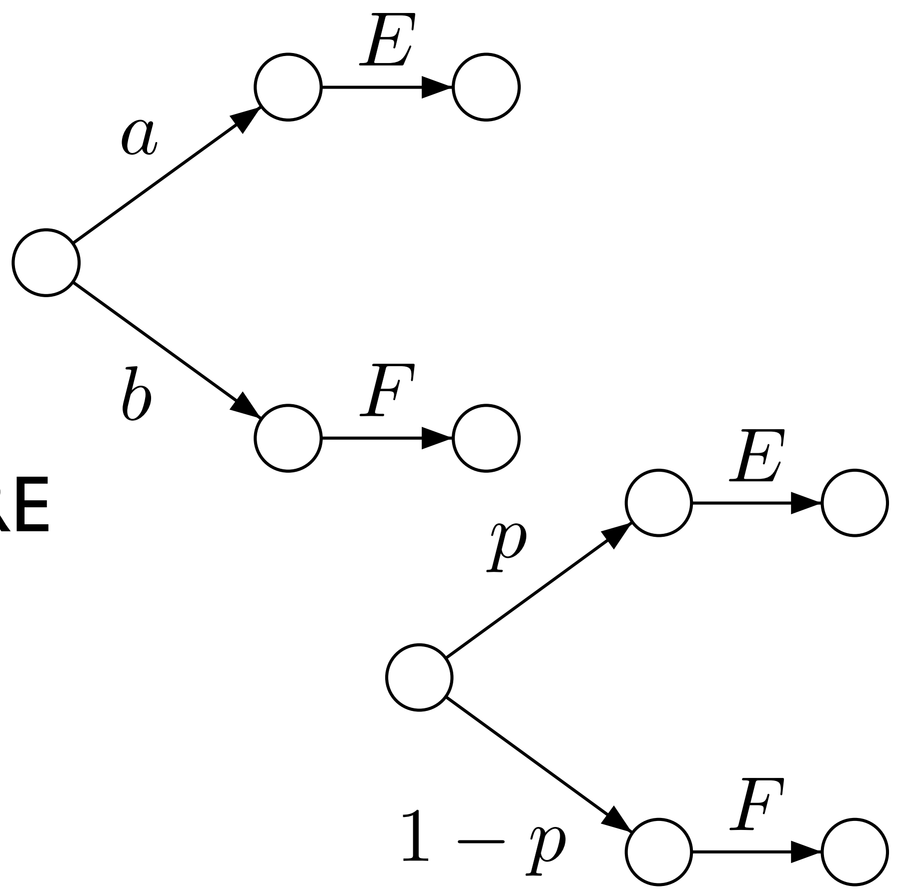

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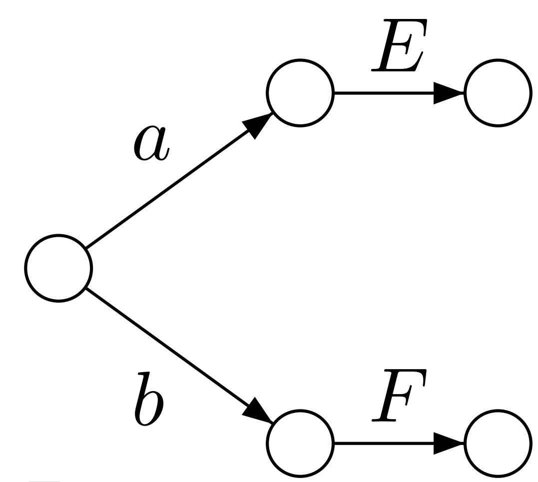

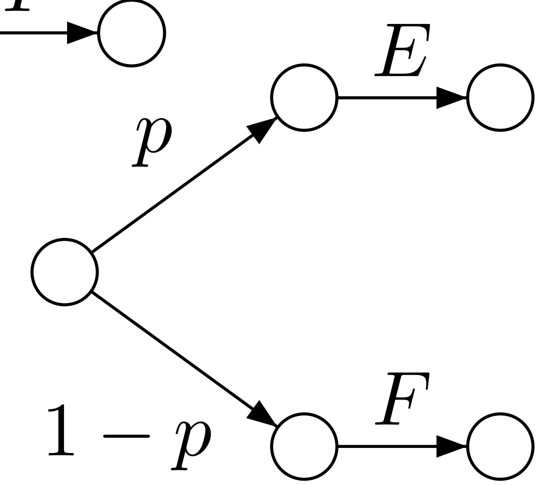

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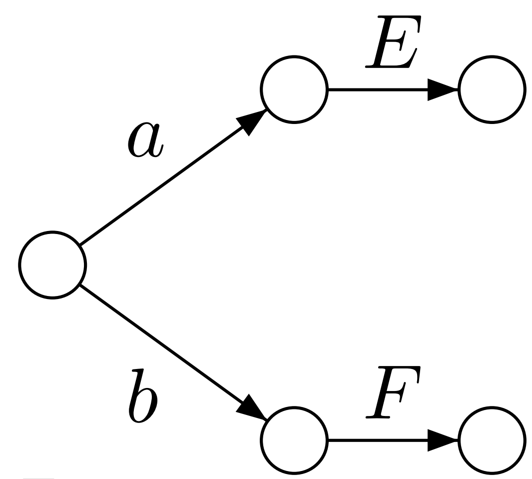

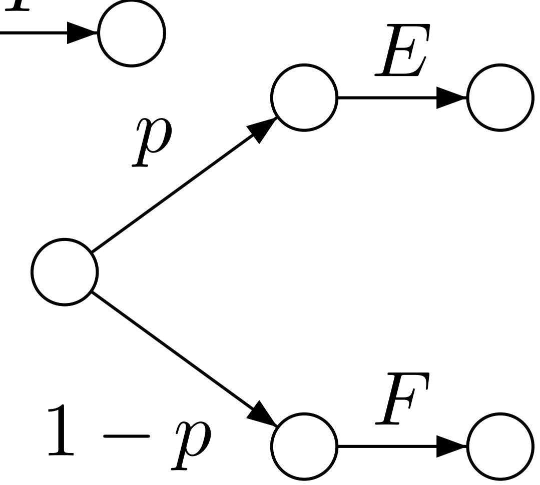

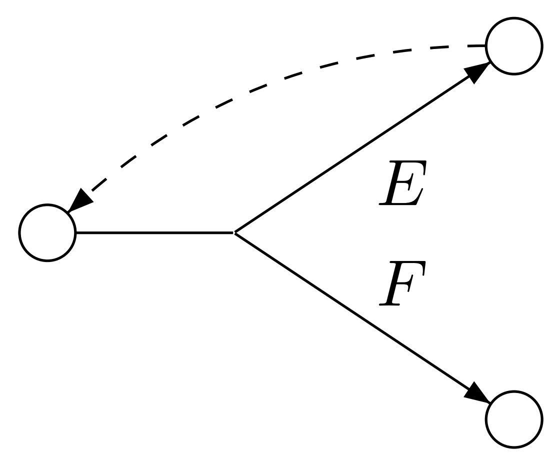

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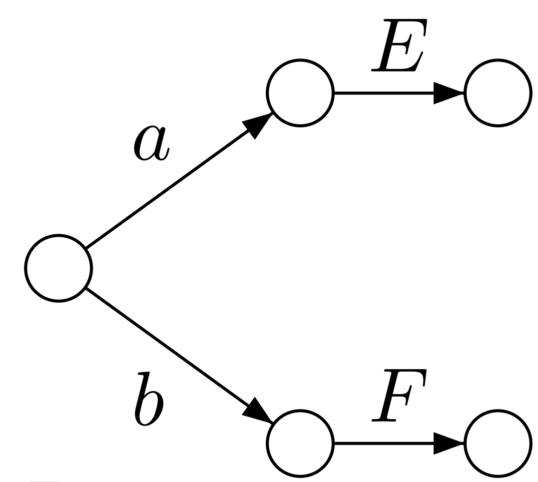

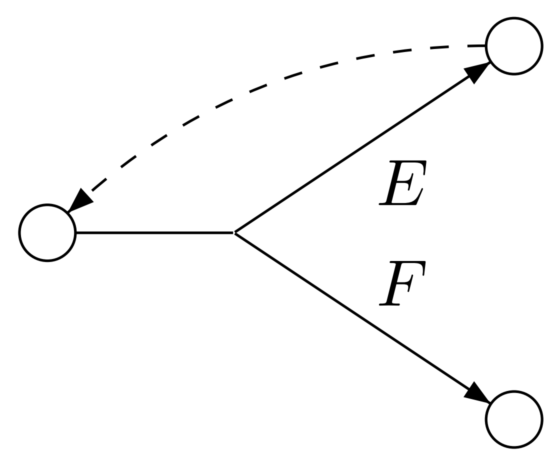

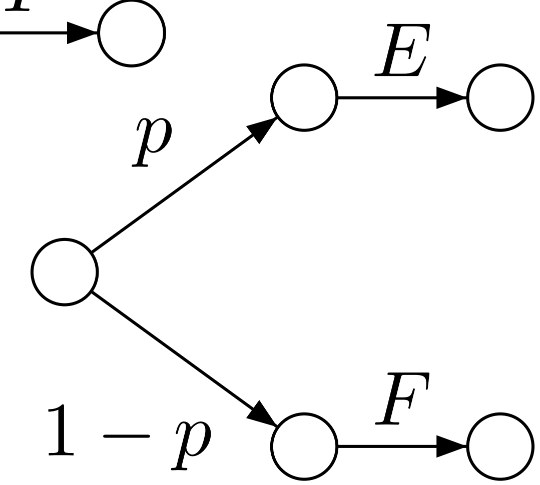

$(a \cdot E)^* \cdot b \cdot F$

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- if E and F are PREs, then $E \cdot F$ is a PRE
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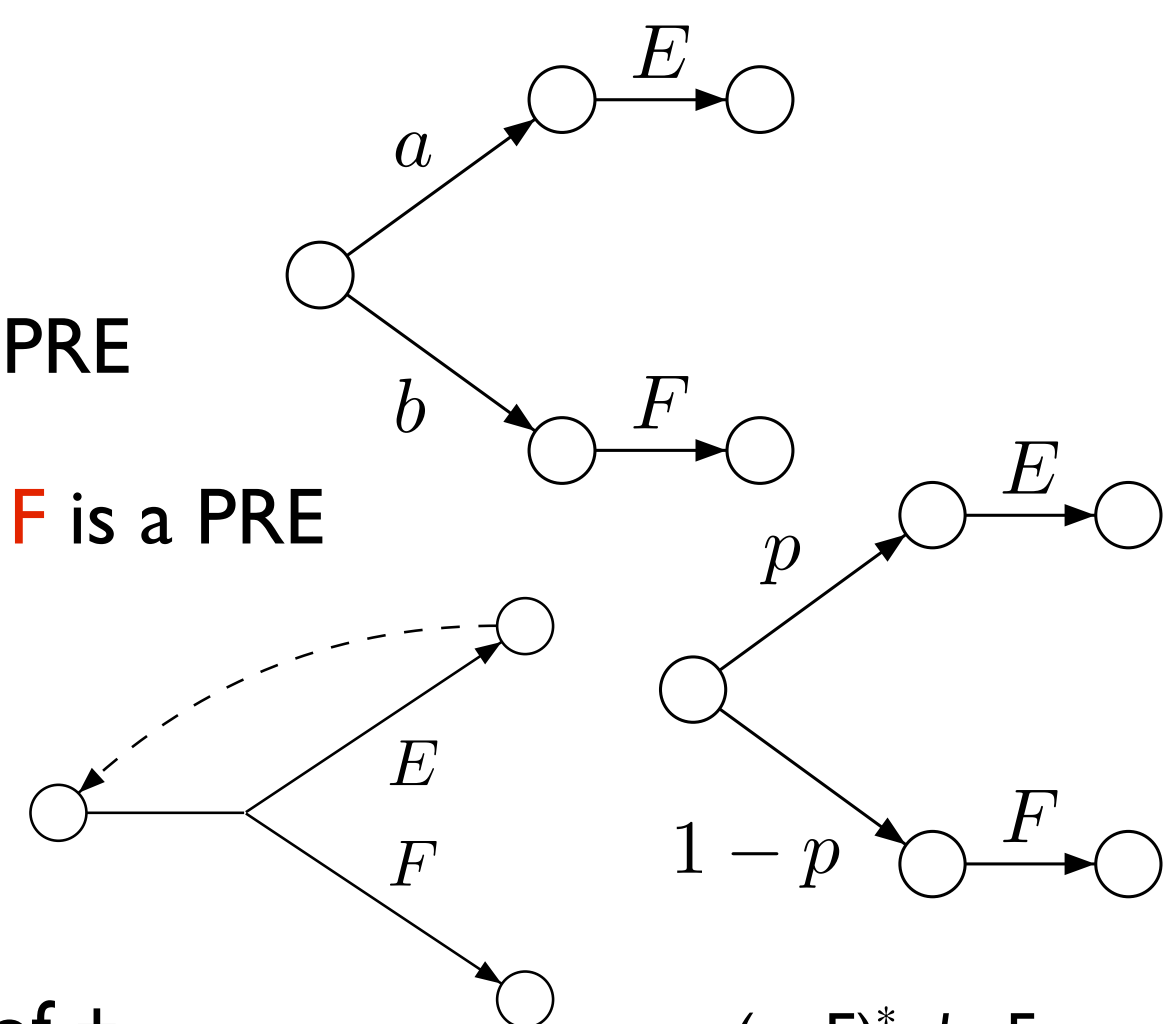
- Closure of PRE under commutativity of +, associativity of + and ·, distributivity of · over +



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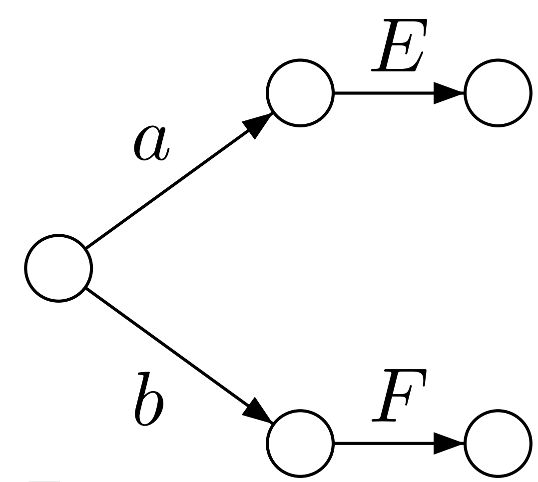
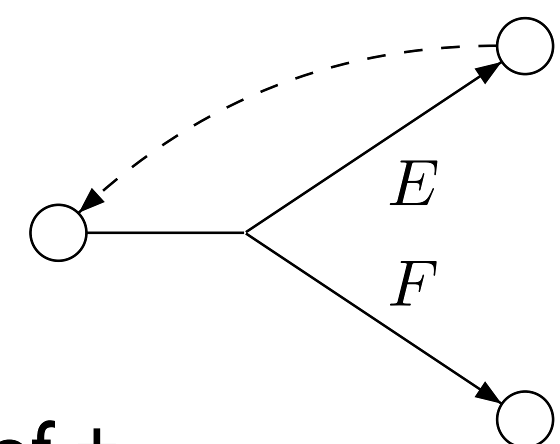
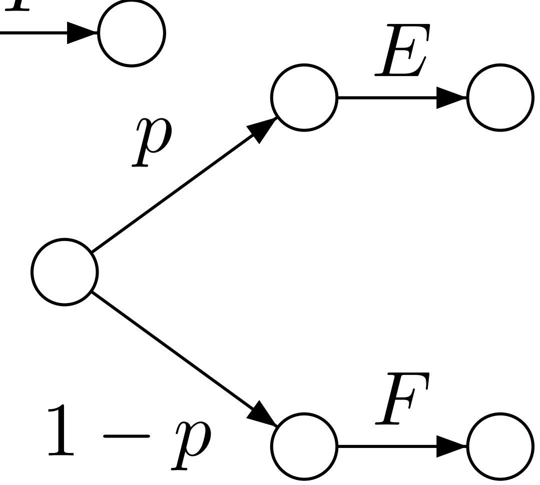
- Closure of PRE under commutativity of +, associativity of + and $\cdot$, distributivity of $\cdot$ over +







$(a \cdot E)^* \cdot b \cdot F$

$(p \cdot E)^* \cdot (1\text{-}p) \cdot F$

Semantics given as a fragment of regular expressions in complete semirings...

$$\mathbb{P}(E \cdot F, u) = \sum_{u=vw} \mathbb{P}(E, v) \times \mathbb{P}(F, w)$$

Example

$$a + b$$

$$a^*b \qquad \frac{1}{2}a + \frac{1}{2}b$$

$$a^*b(\frac{1}{2}a + \frac{1}{2}b)$$

$$a^*b\frac{1}{2}a + a^*b\frac{1}{2}b$$

$$(a^*b\frac{1}{2}a)^*a^*b\frac{1}{2}b$$

Example

Deterministic choice

$$a + b$$

$$\downarrow$$

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$$\searrow \quad \swarrow$$

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Example

Deterministic choice

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Star rule

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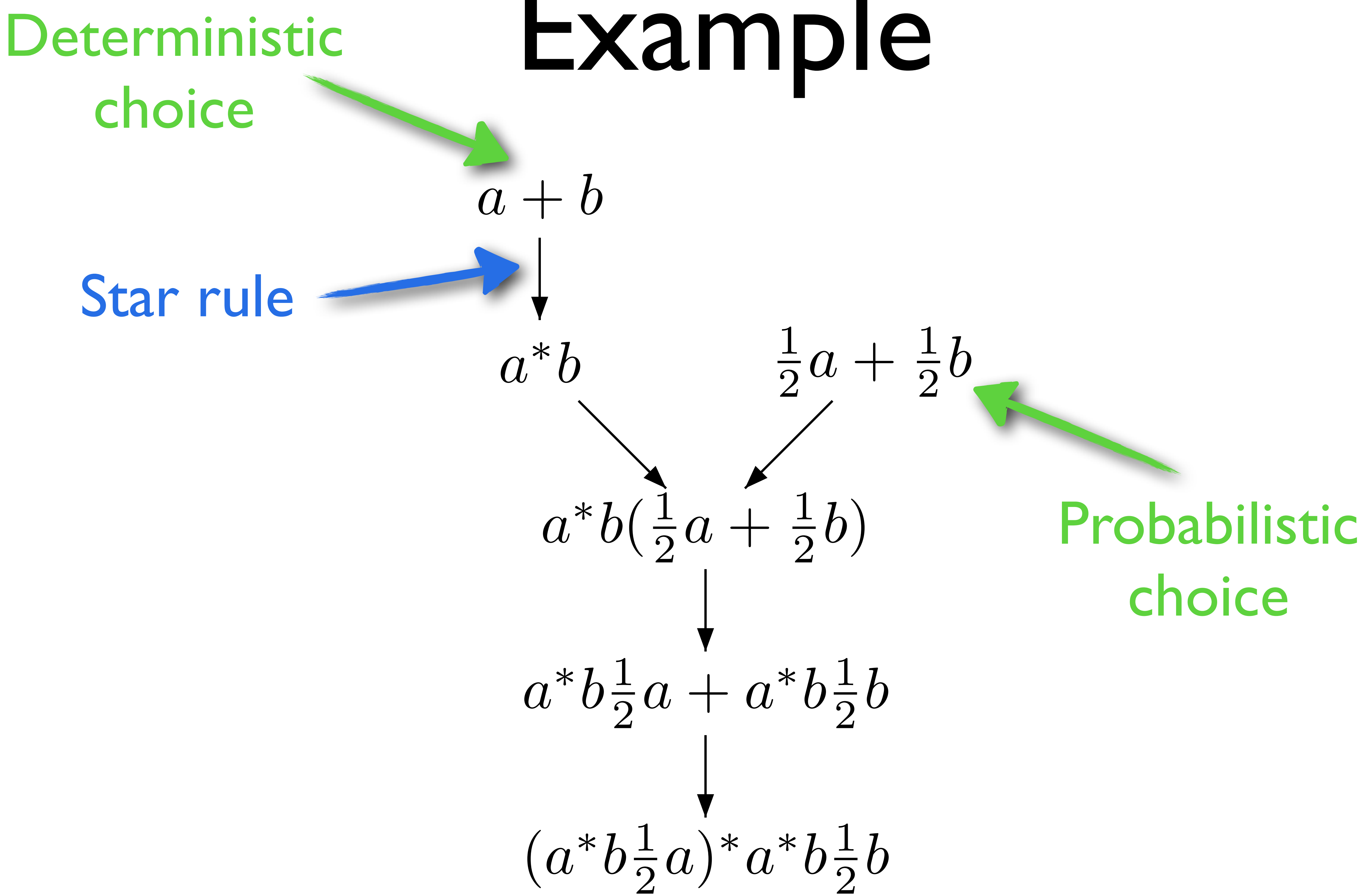

Deterministic choice
Star rule
Probabilistic choice
$a + b$
a^*b
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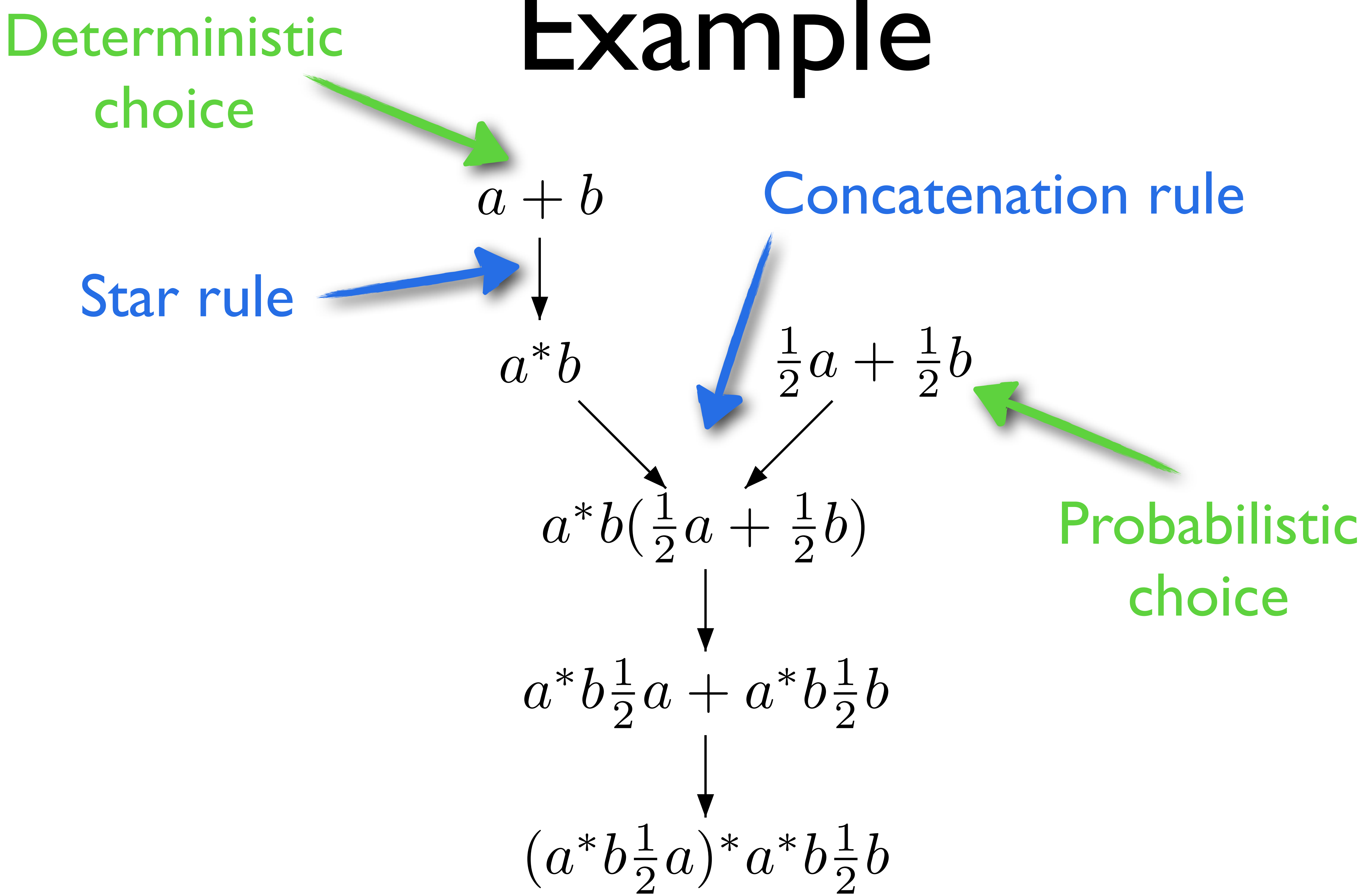
Deterministic choice
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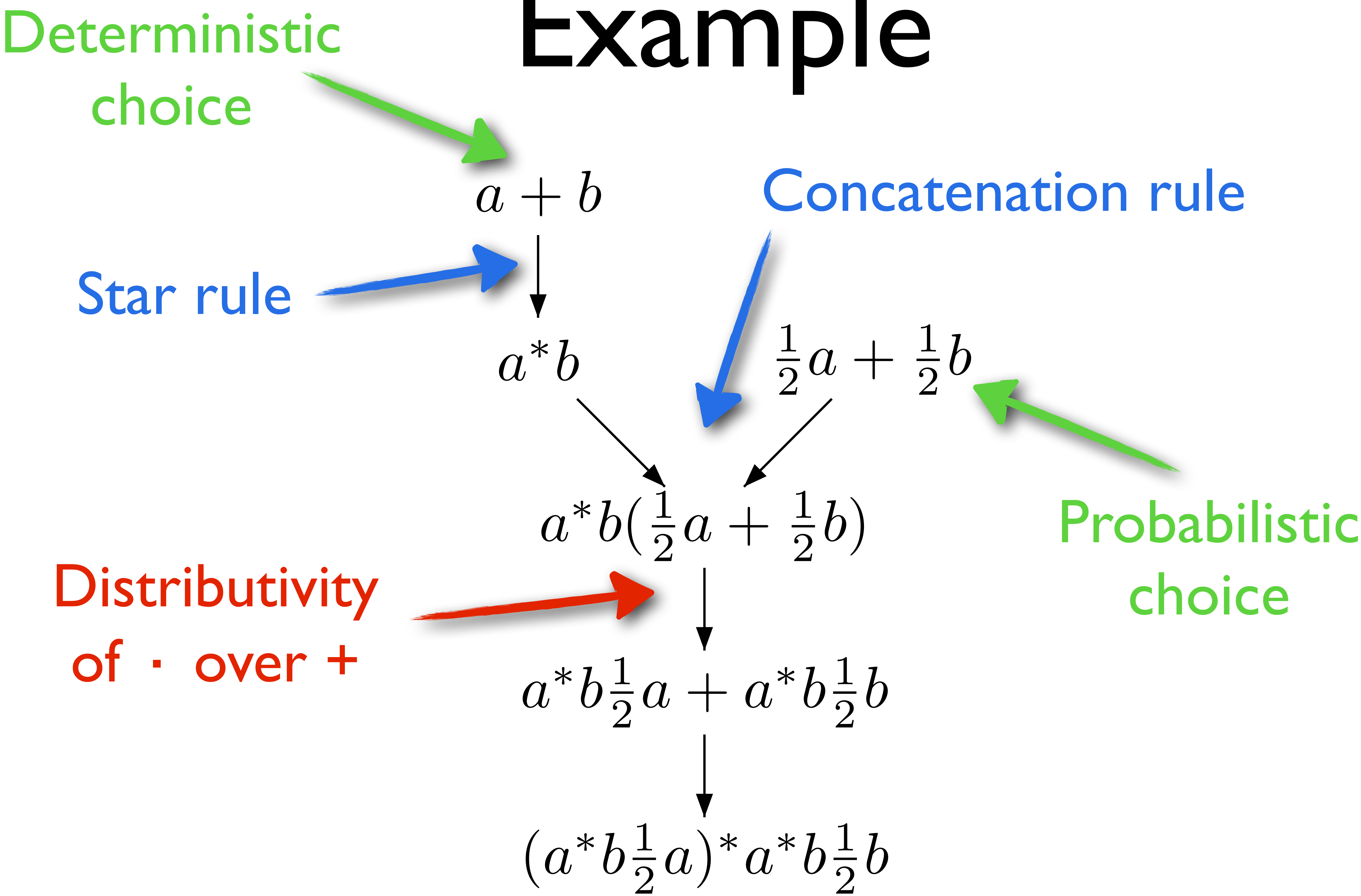

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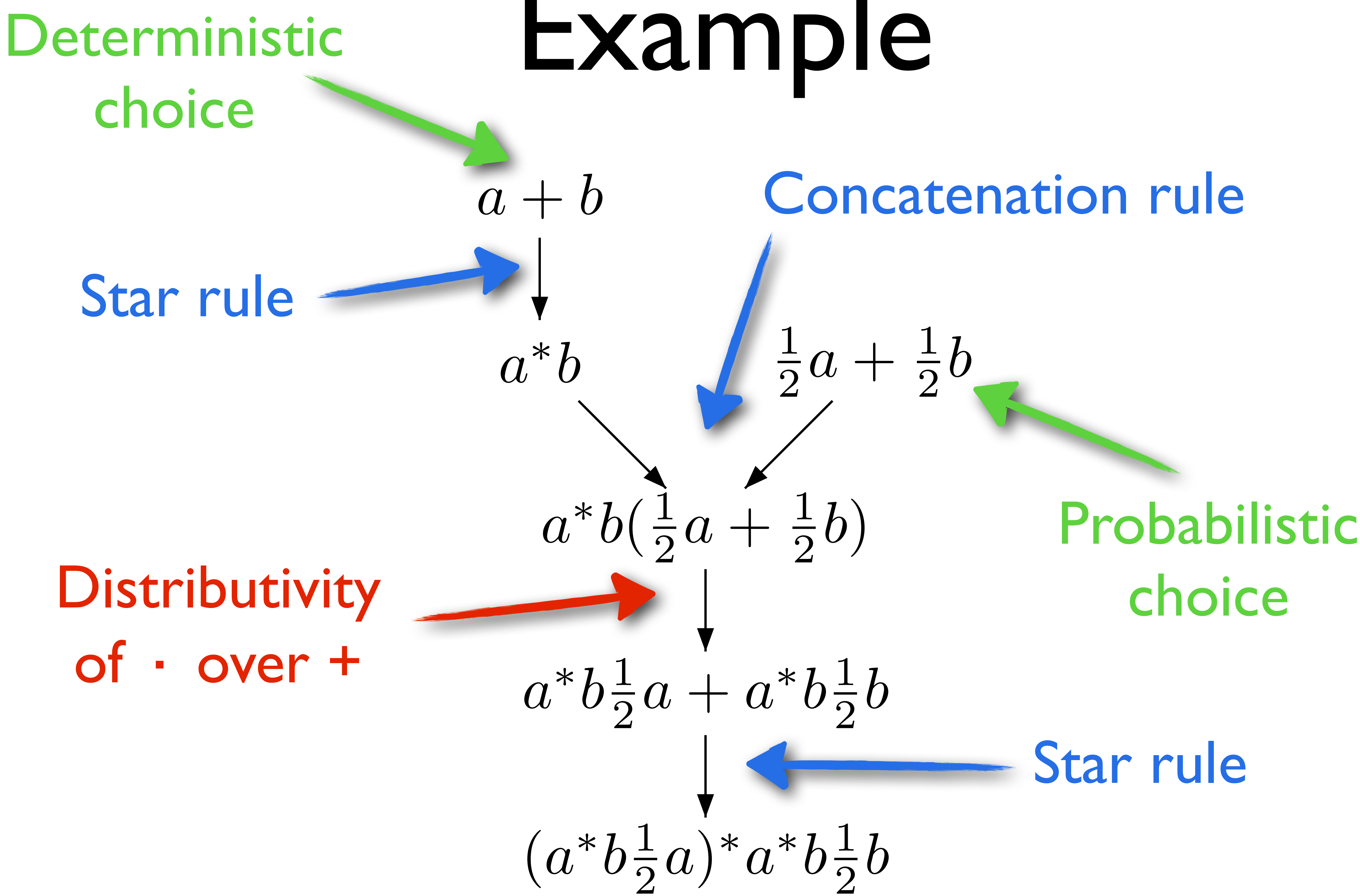
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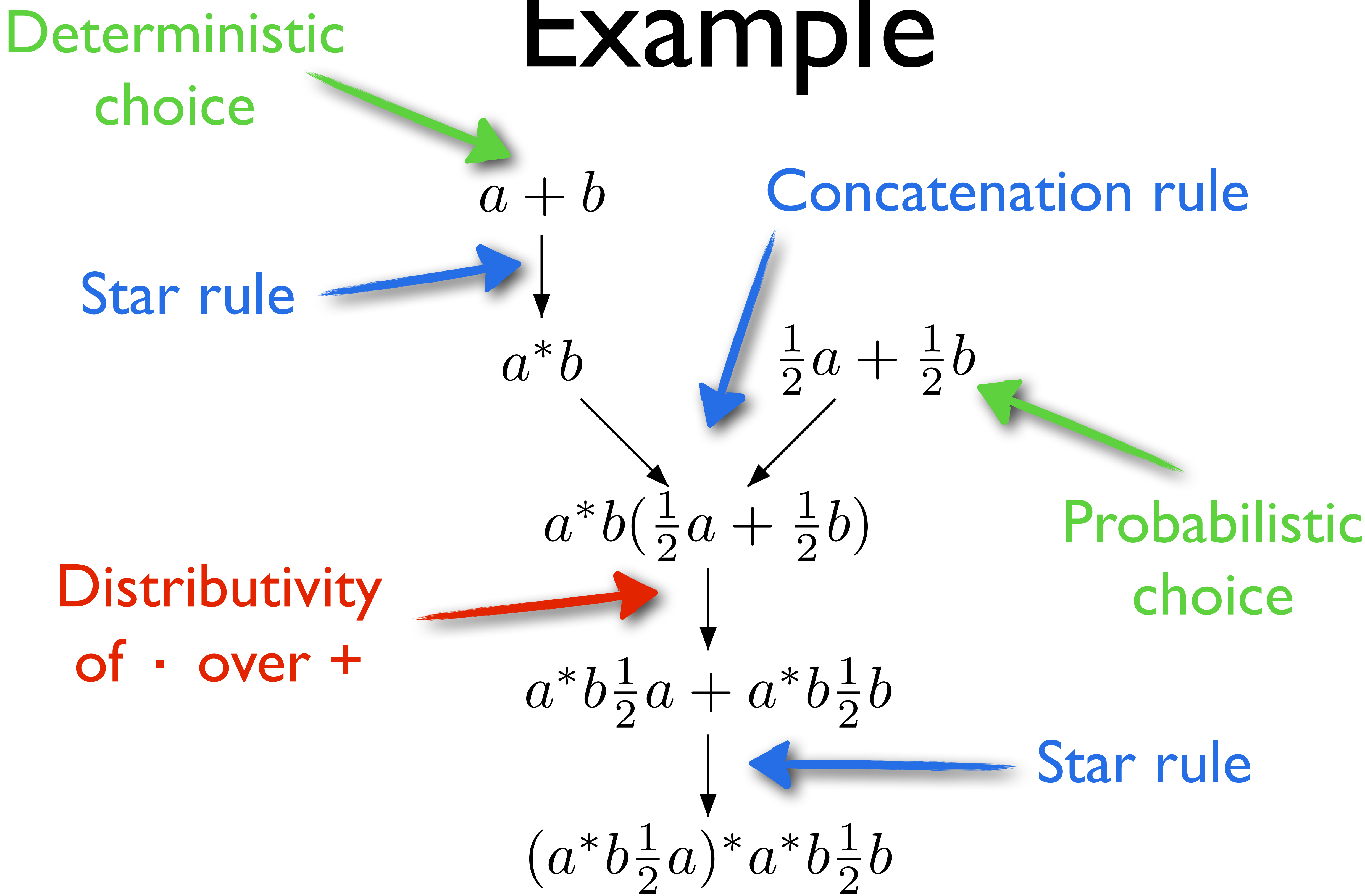

The choice in the star is made *far* from the beginning...

Probabilistic Kleene-Schützenberger Theorem

- Every PRE can be translated into an equivalent Probabilistic automaton.
- Every Probabilistic automaton can be denoted by an equivalent PRE.

From Automata to Expressions

- Usual procedures (Brozozwski-McCluskey, elimination, McNaughton-Yamada...) keeping probabilistic constraints in mind
- Requires to prove some (useful) properties of PREs, e.g., if E+F and G are PREs, then E+F · G is a PRE

[1] J. A. Brzozowski and E. J. McCluskey (1963). Signal Flow Graph Techniques for Sequential Circuit State Diagrams. IEEE Trans. on Electronic Computers 12.

From Expressions to Automata

[1] V. M. Glushkov (1961). The abstract theory of automata. Russian Math. Surveys 16.
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From Expressions to Automata

- We have to prove closure properties of Probabilistic Automata (for example constructing standard automata, [1,2])

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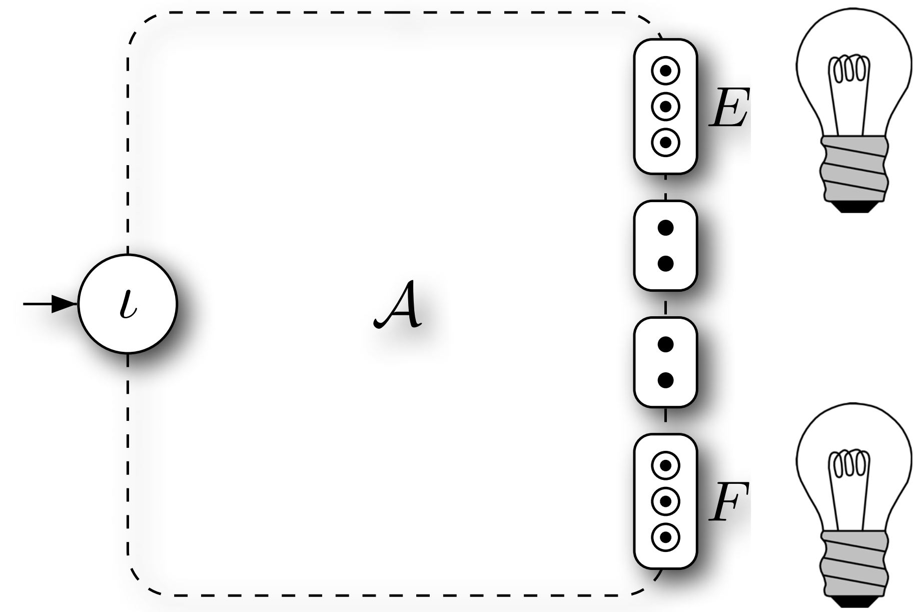

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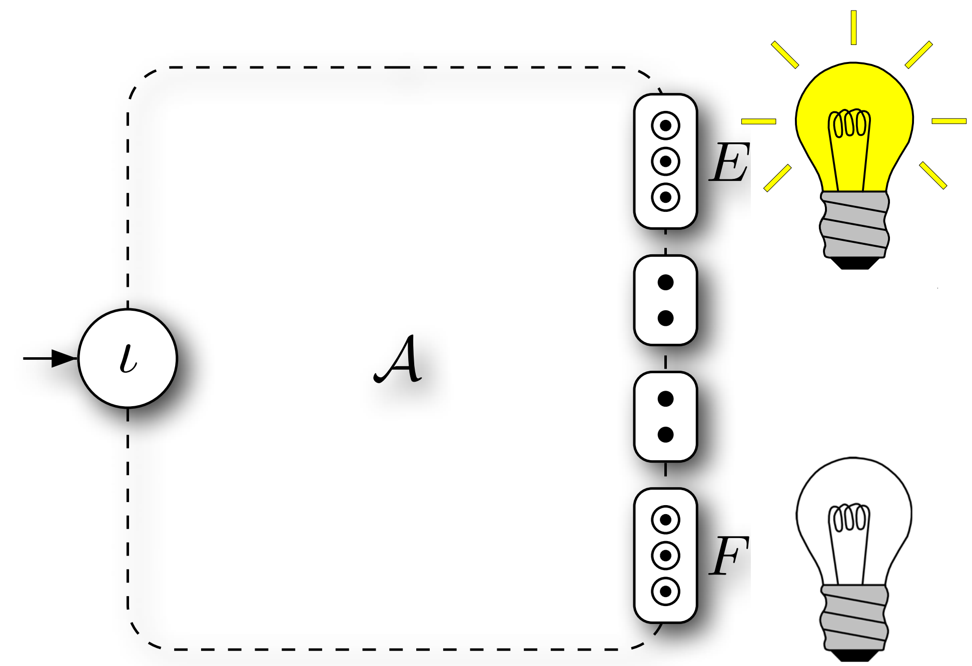

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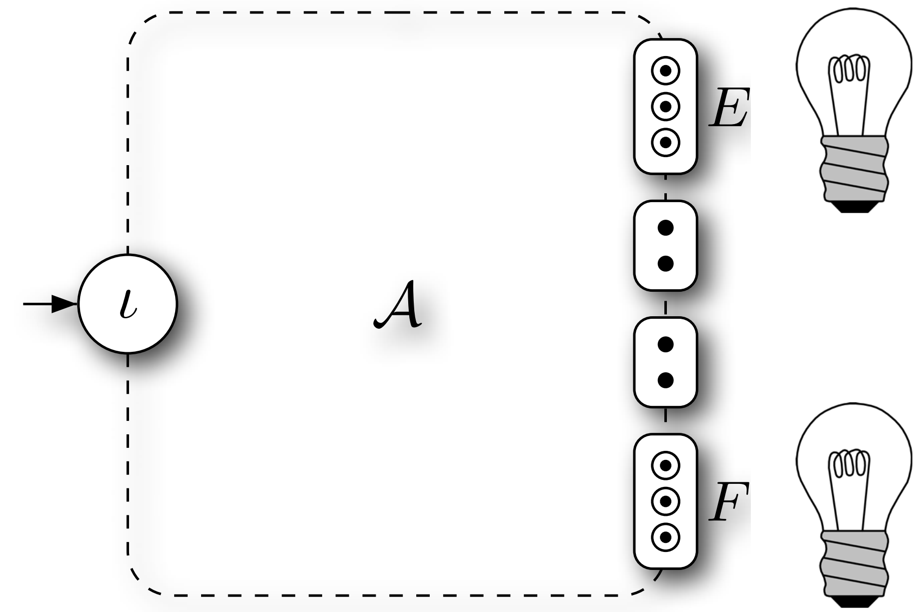

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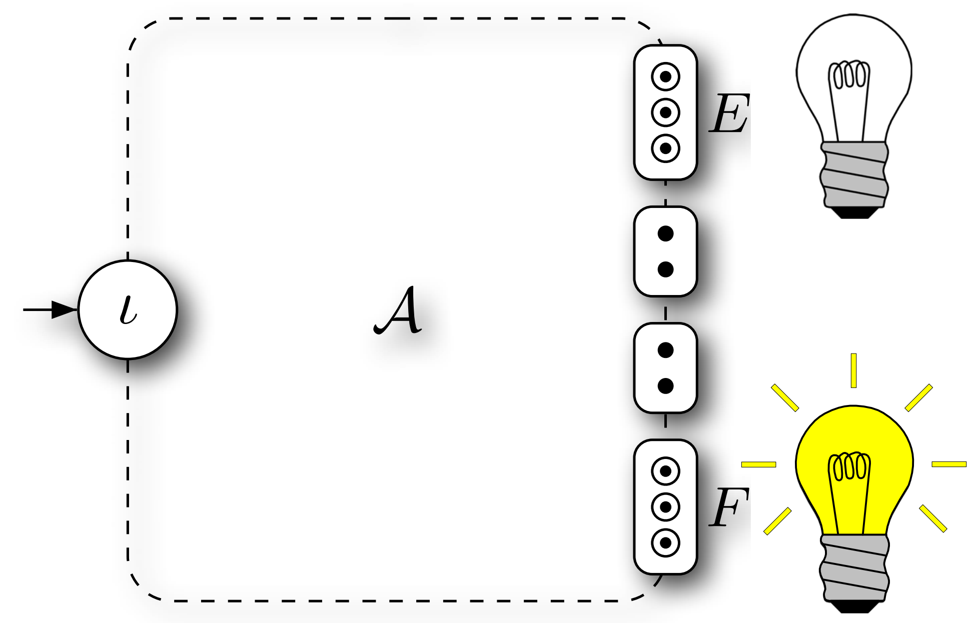

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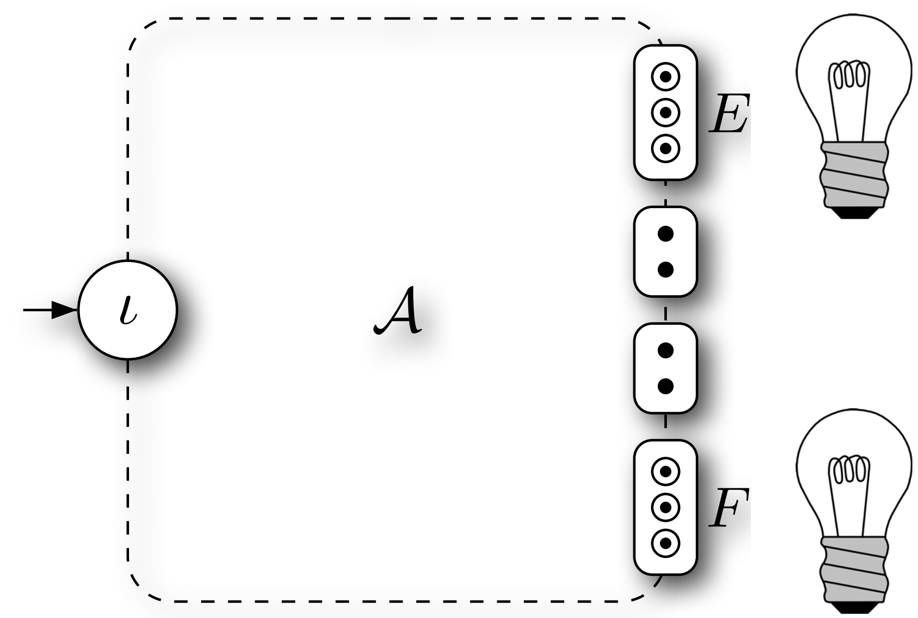

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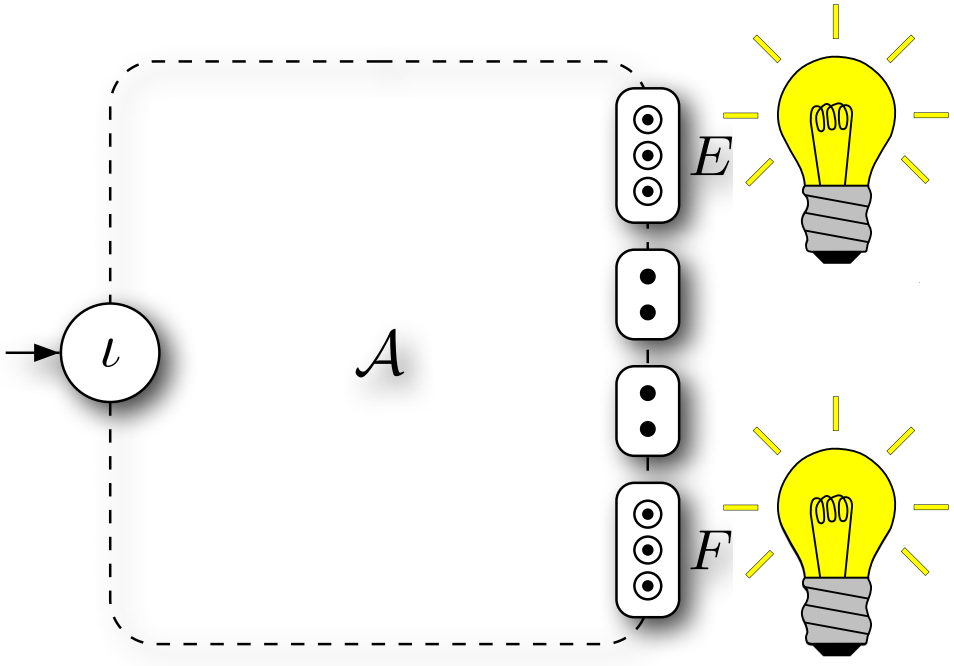

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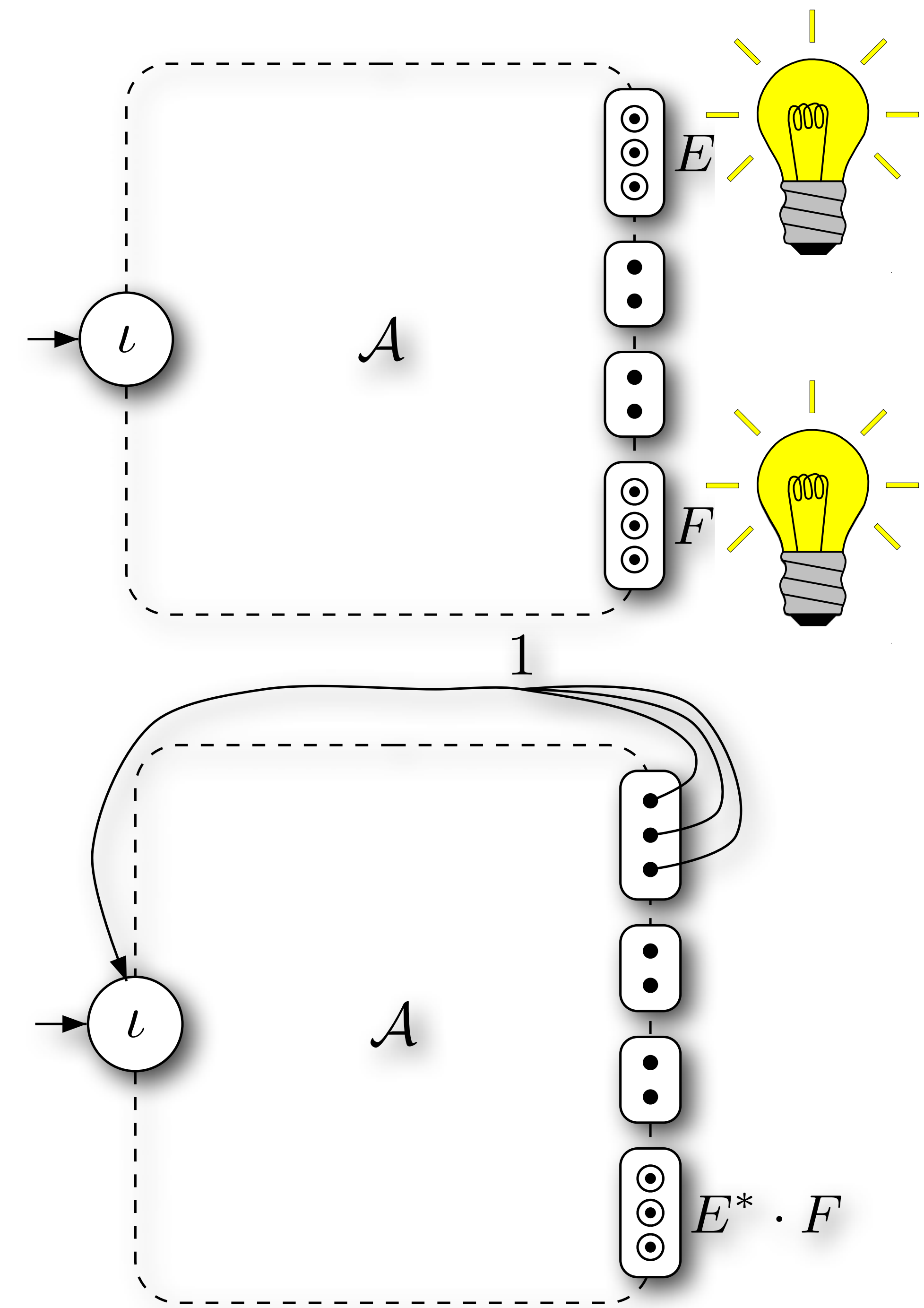

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Corollaries

- Equivalence problem for PREs is decidable: given PREs E and F, does they generate the same semantics? (translation into automata [1])
- Threshold problem for PREs is undecidable: given a PRE E and a threshold *s*, is there a word *w* which is mapped by to a probability greater than *s*? (by reduction to automata [2])

[1] M.-P. Schützenberger (1961). On the Definition of a Family of Automata. Information and Control.
[2] A. Paz. (1971). Introduction to probabilistic automata. Academic Press,

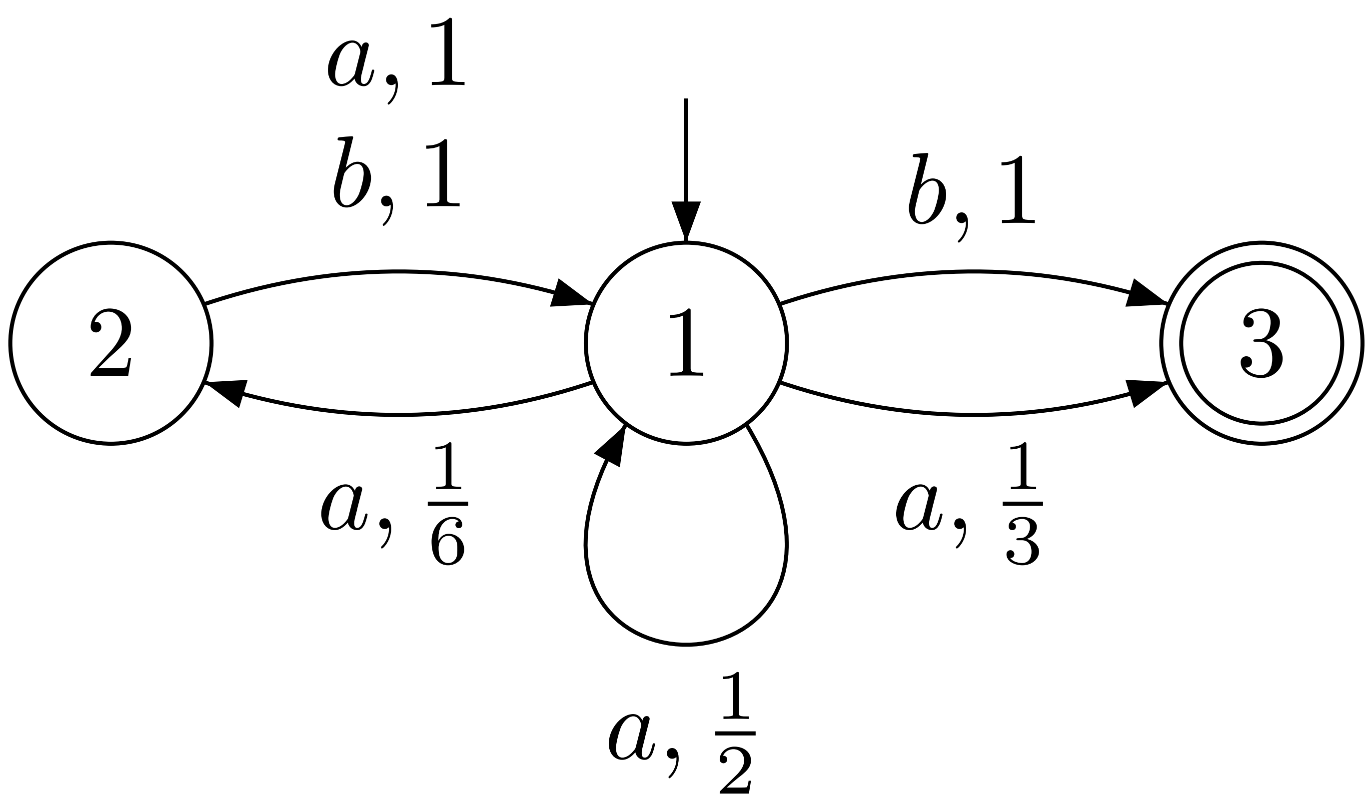
Summary and Future Works

- General Kleene-Schützenberger theorems for Probabilistic models (classical, extended to two-way automata, pebble automata in full paper [1])
- Study of Probabilistic Expressions and their extensions permits us to better understand which behavior Probabilistic Automata can generate
- In [2], we proved that Weighted Automata (with two-way and pebbles) can be evaluated efficiently
- Future work: get logical formalisms generating the same expressivity, and implement quick algorithms to perform translation from PREs to PAs (as there are some for weighted automata, see [2,3] e.g.)

[1] B. Bollig, P. Gastin, B. M. and M. Zeitoun. (2012). A Probabilistic Kleene Theorem. In Proceedings of ATVA'12.
[2] P. Gastin and B. M. (2006). Adding Pebbles to Weighted Automata. In Proceedings of CIAA'12.
[3] C. Allauzen, and M., Mohri, (2006). A Unified Construction of the Glushkov, Follow, and Antimirov Automata. In Proceedings of MFCS'06

Automata Model

Usual Rabin automata...



$$\mathcal{A} = (Q, \iota, Acc, \mathbb{P})$$

$$\mathbb{P} : Q \times \Sigma \times Q \to [0, 1]$$

$$Acc(q) + \sum_{q' \in Q} \mathbb{P}(q, a, q') \leq 1 \text{ for all } (q, a) \in Q \times A$$

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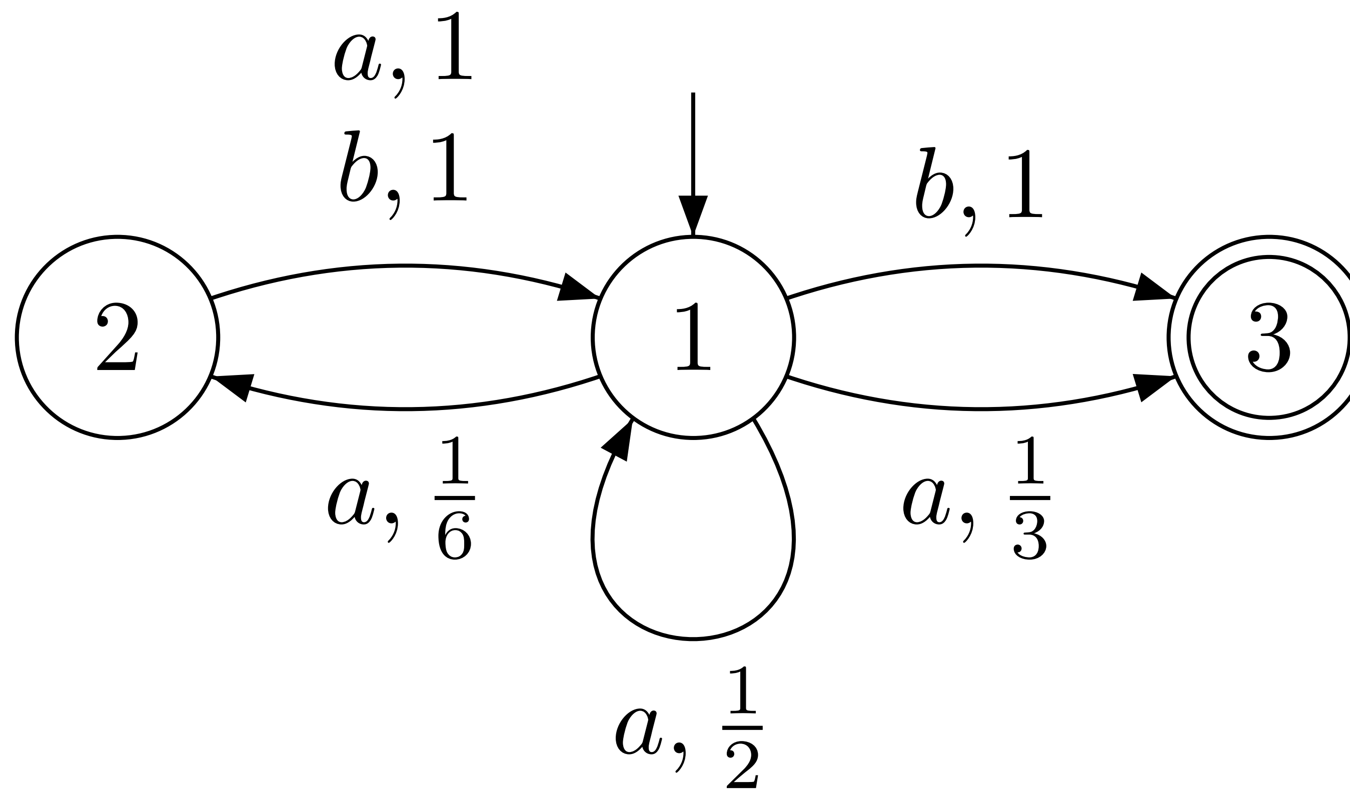

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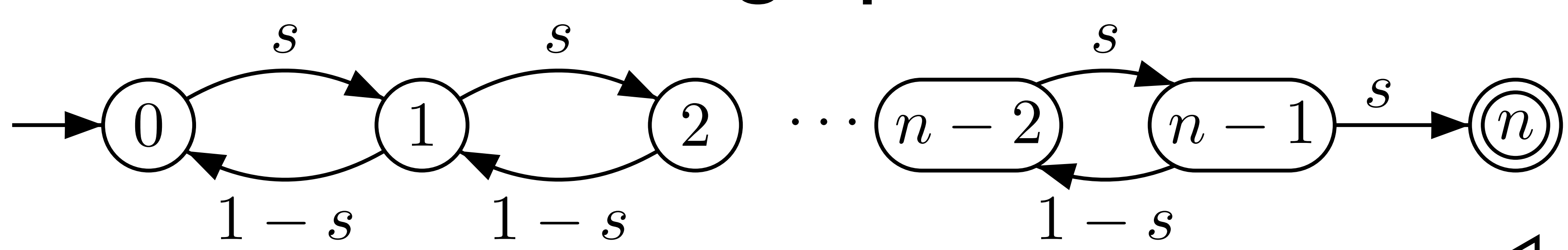
GOAL: Remove all trace of non-determinism

- seems to be a strong restriction

+ indeed we can drop it using a right marker $\triangleleft$ in words

2-way Probabilistic Expressions

Random Walk over a finite linear graph



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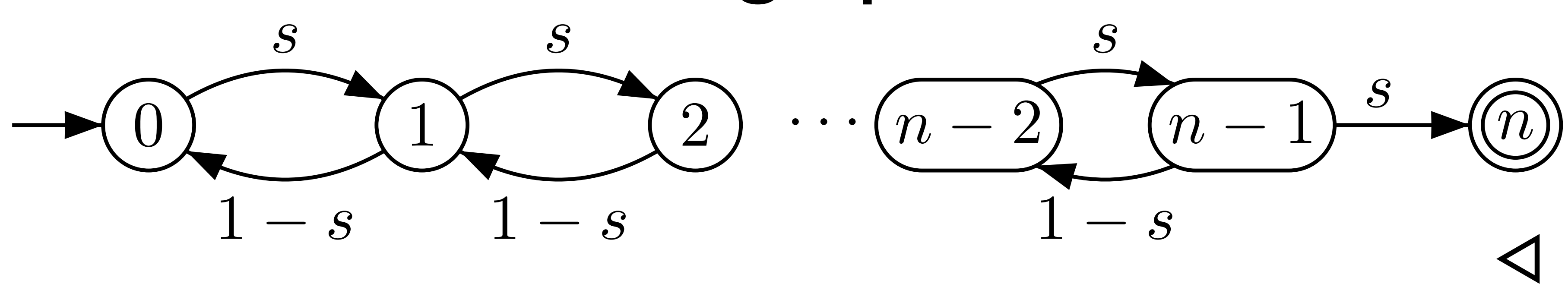
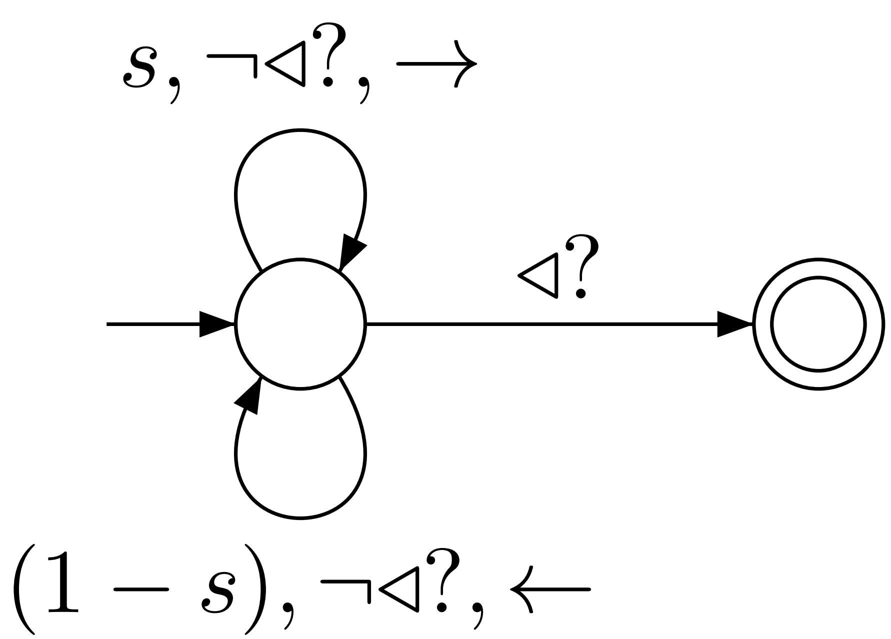


Expressible with
Probabilistic 2-way Automata



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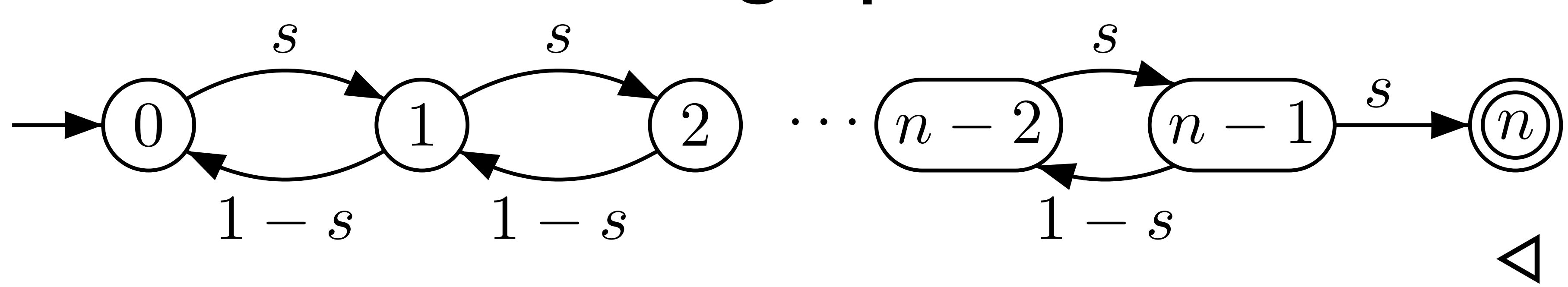

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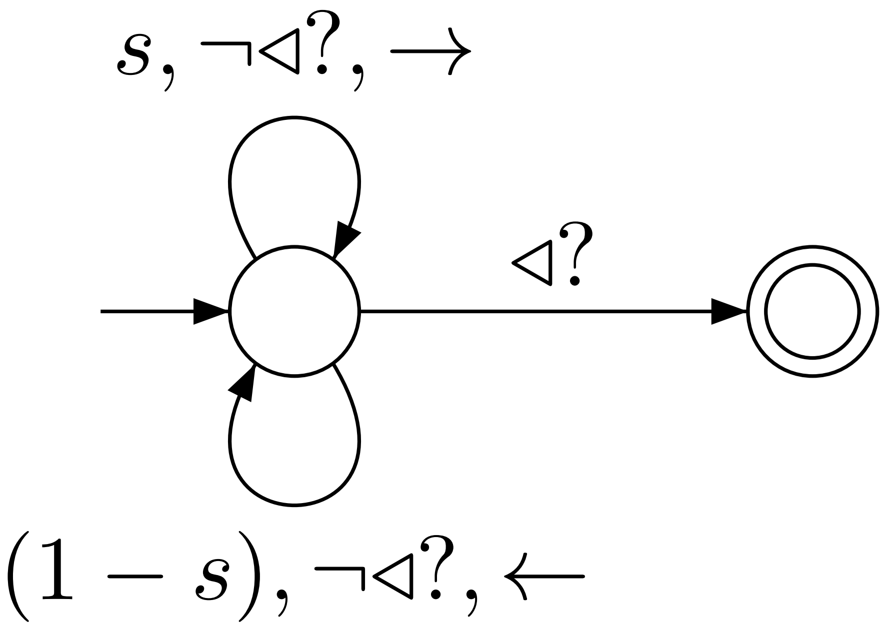

Expressible with
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$$E = \left(\neg\triangleleft?\, s\!\rightarrow + \neg\triangleleft?(1-s)\!\leftarrow\right)^{*}\triangleleft?$$

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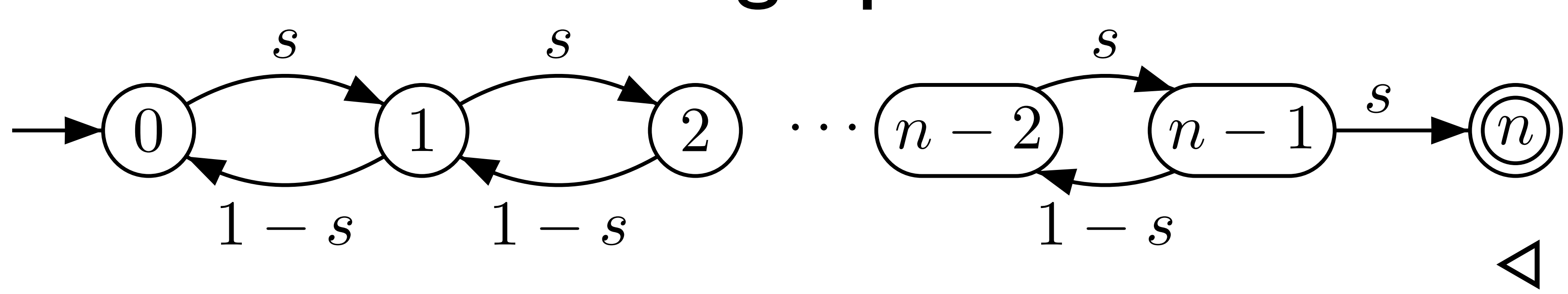

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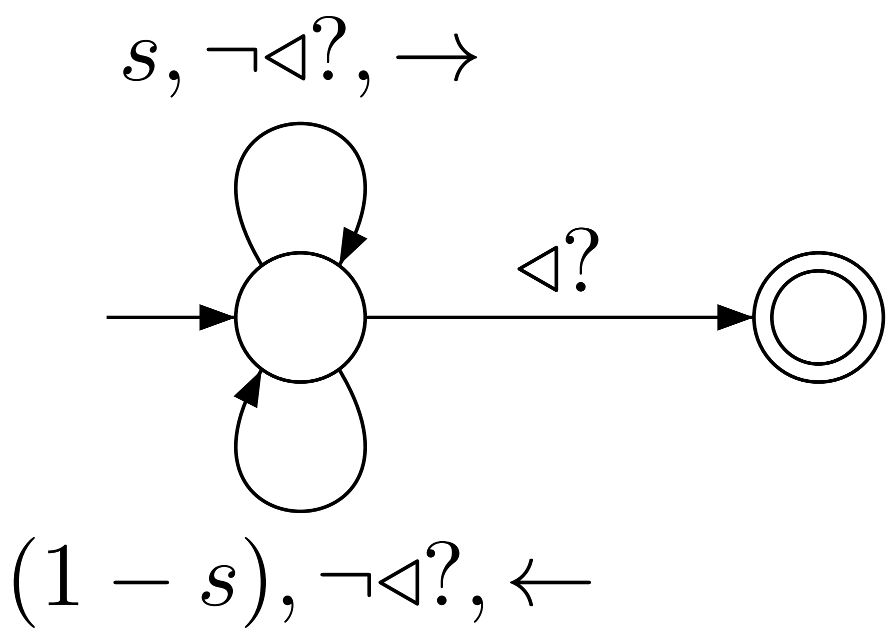

Expressible with
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$$E = \left(\neg\triangleleft? s \rightarrow + \neg\triangleleft?(1-s)\leftarrow\right)^* \triangleleft?$$

Not expressible with
Probabilistic Expressions / Probabilistic Automata

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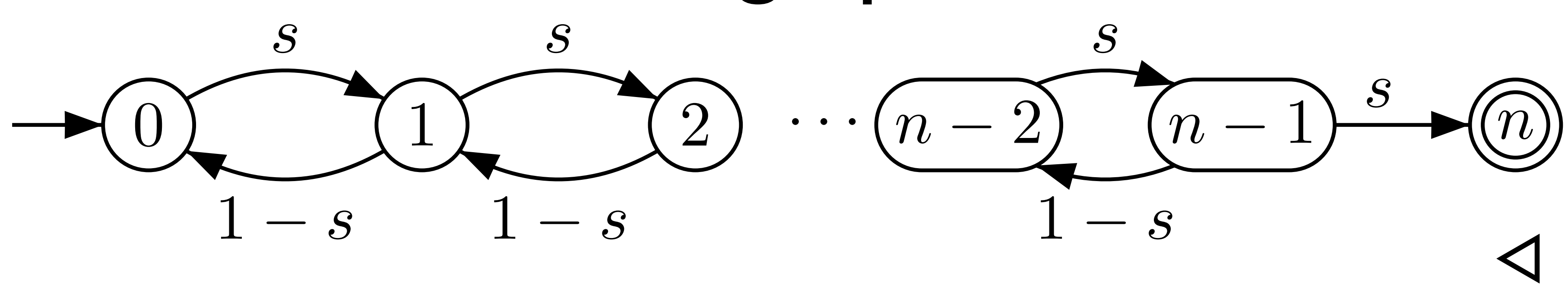

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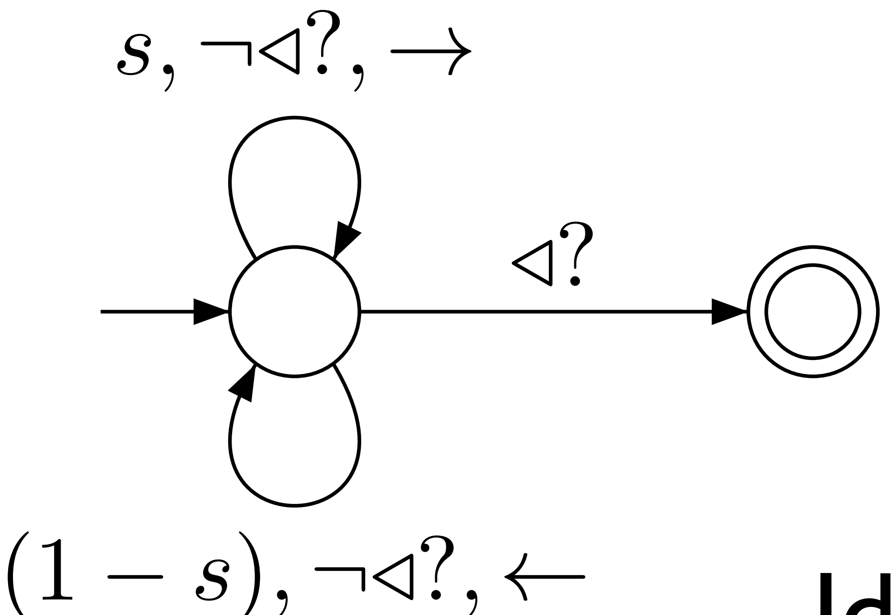

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Idea: replace every letter *a* by a test *a?*
followed by a move (either → or ←)

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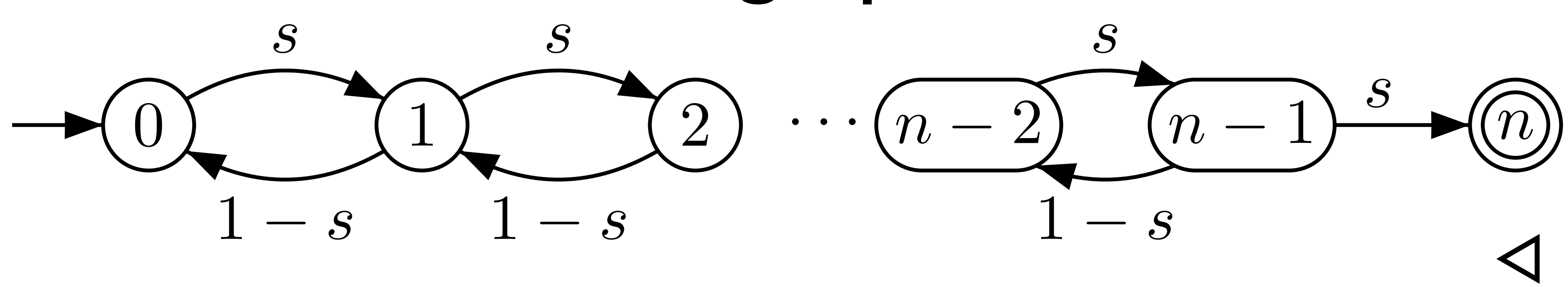

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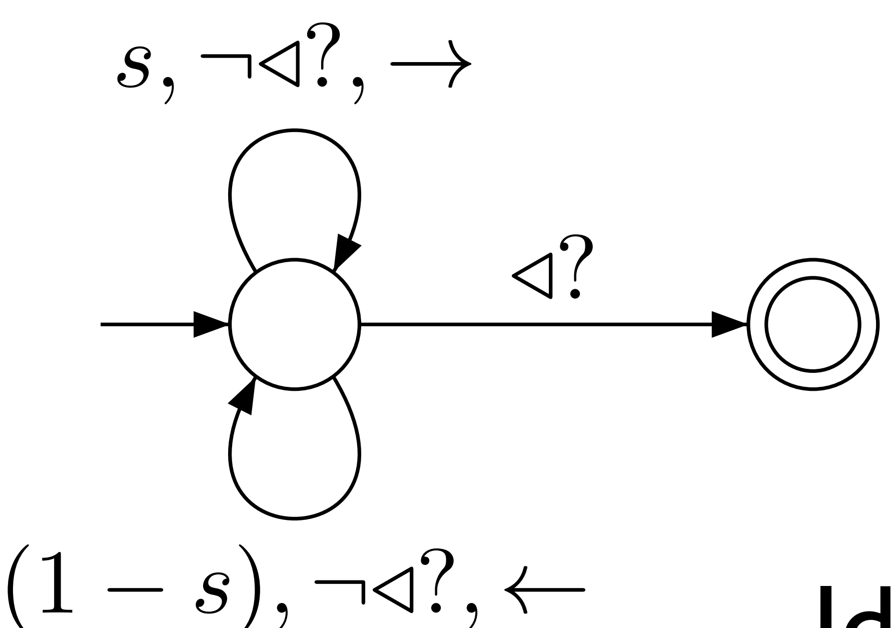

Expressible with
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$$E = \left(\neg\triangleleft ?\, s\!\rightarrow + \neg\triangleleft ?\,(1-s)\!\leftarrow\right)^{*} \triangleleft ?$$

Not expressible with
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Idea: replace every letter *a* by a test *a?*
followed by a move (either → or ←)

Expressiveness result still holds!

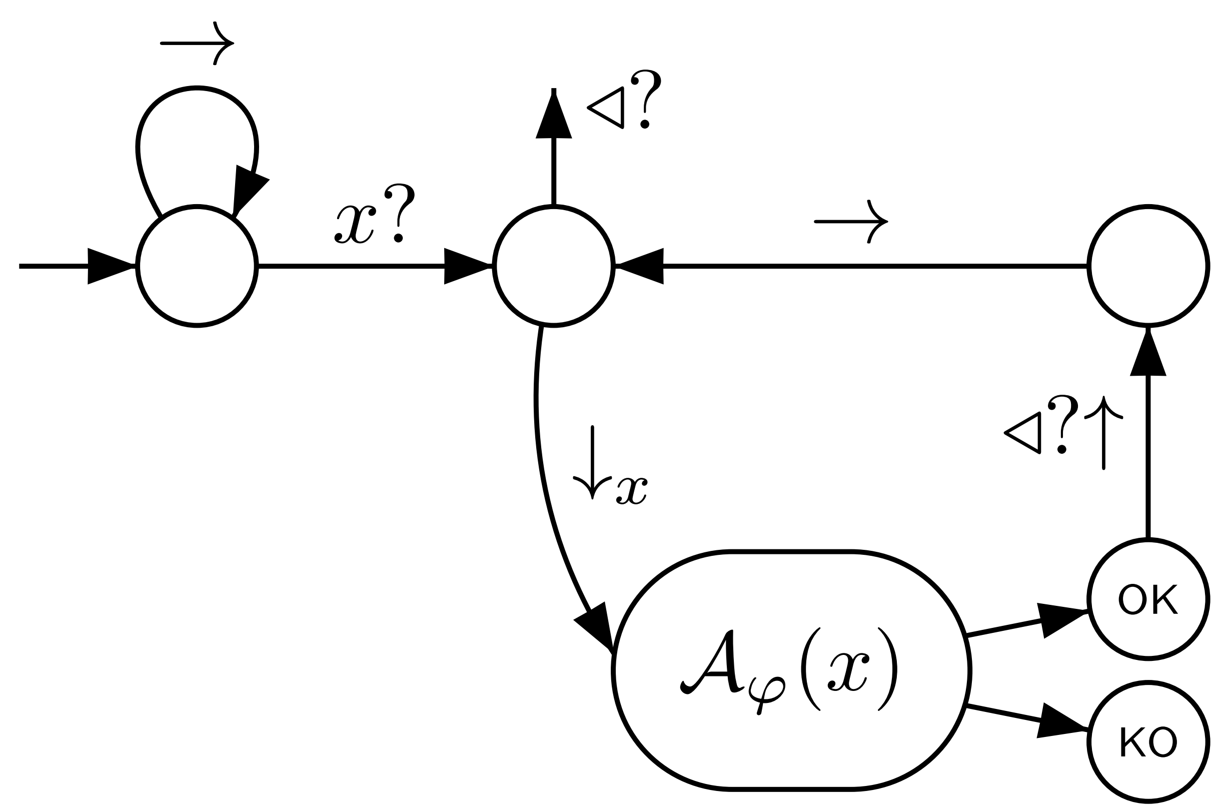
Adding Pebbles: pLTL

Each LTL formula φ has an implicit free variable *x* denoting the position where the formula is evaluated. We use a pebble to mark this position.

Let *P*(φ, *u*, *i*) denote the probability that φ holds on word *u* at position *i*.

$$\mathbb{P}(\mathsf{G}\,\varphi, u, i) \quad = \quad \prod_{j \geq i} \mathbb{P}(\varphi, u, j)$$

$$\mathcal{A}_{\mathsf{G}\,\varphi}(x) \quad =$$



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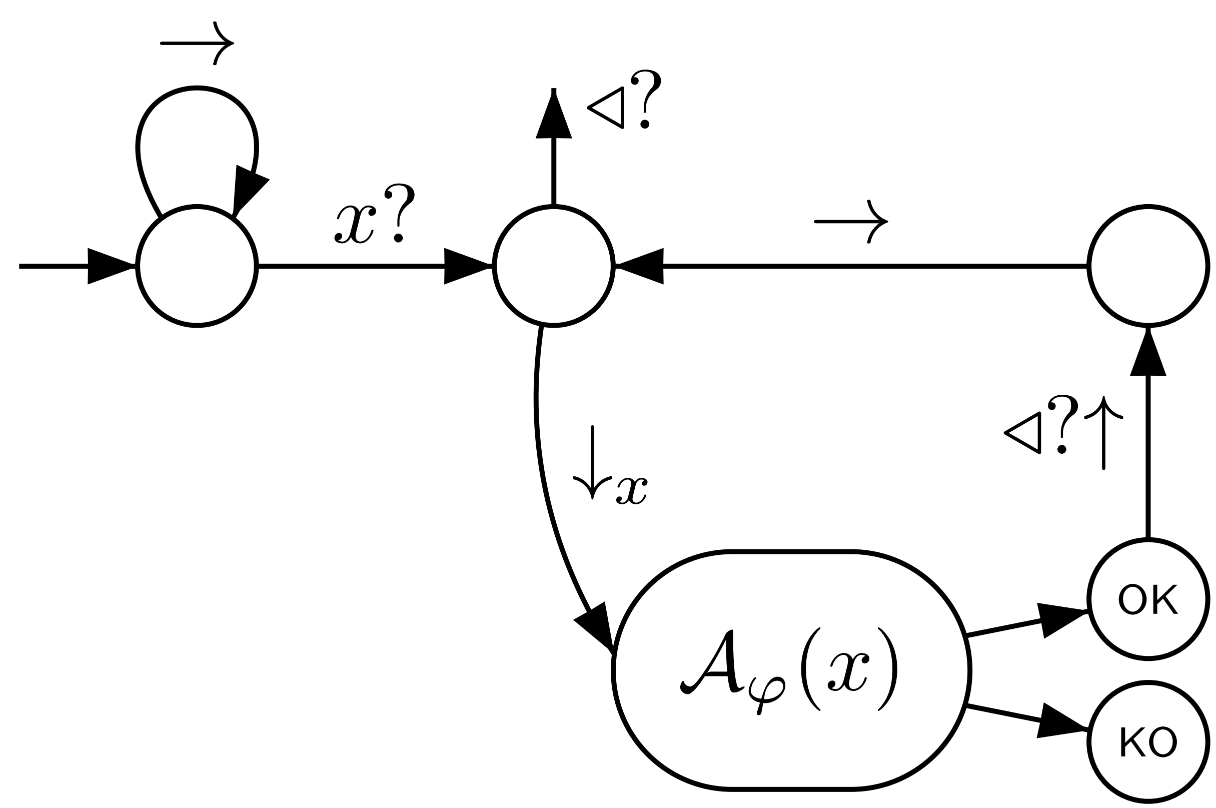

$$E_{\mathsf{G}\,\varphi}(x) = \triangleright ? \rightarrow^* x?\big((x!E_\varphi(x))\rightarrow\big)^* \triangleleft ?$$

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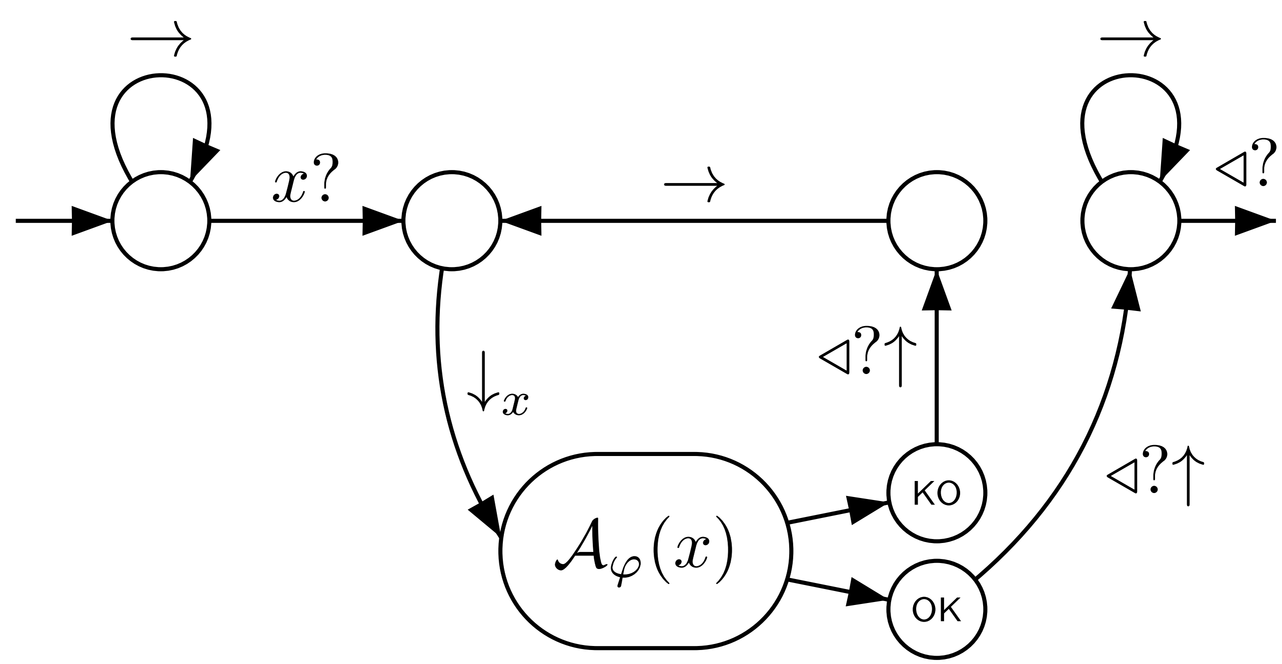
$$\begin{aligned}\mathbb{P}(\mathsf{F}\,\varphi, u, i) &= \mathbb{P}(\varphi, u, i) + (1 - \mathbb{P}(\varphi, u, i)) \times \mathbb{P}(\mathsf{F}\,\varphi, u, i+1) \\ &= \sum_{j \geq i} \left(\prod_{i \leq k < j} \mathbb{P}(\neg\varphi, u, k) \right) \times \mathbb{P}(\varphi, u, j)\end{aligned}$$

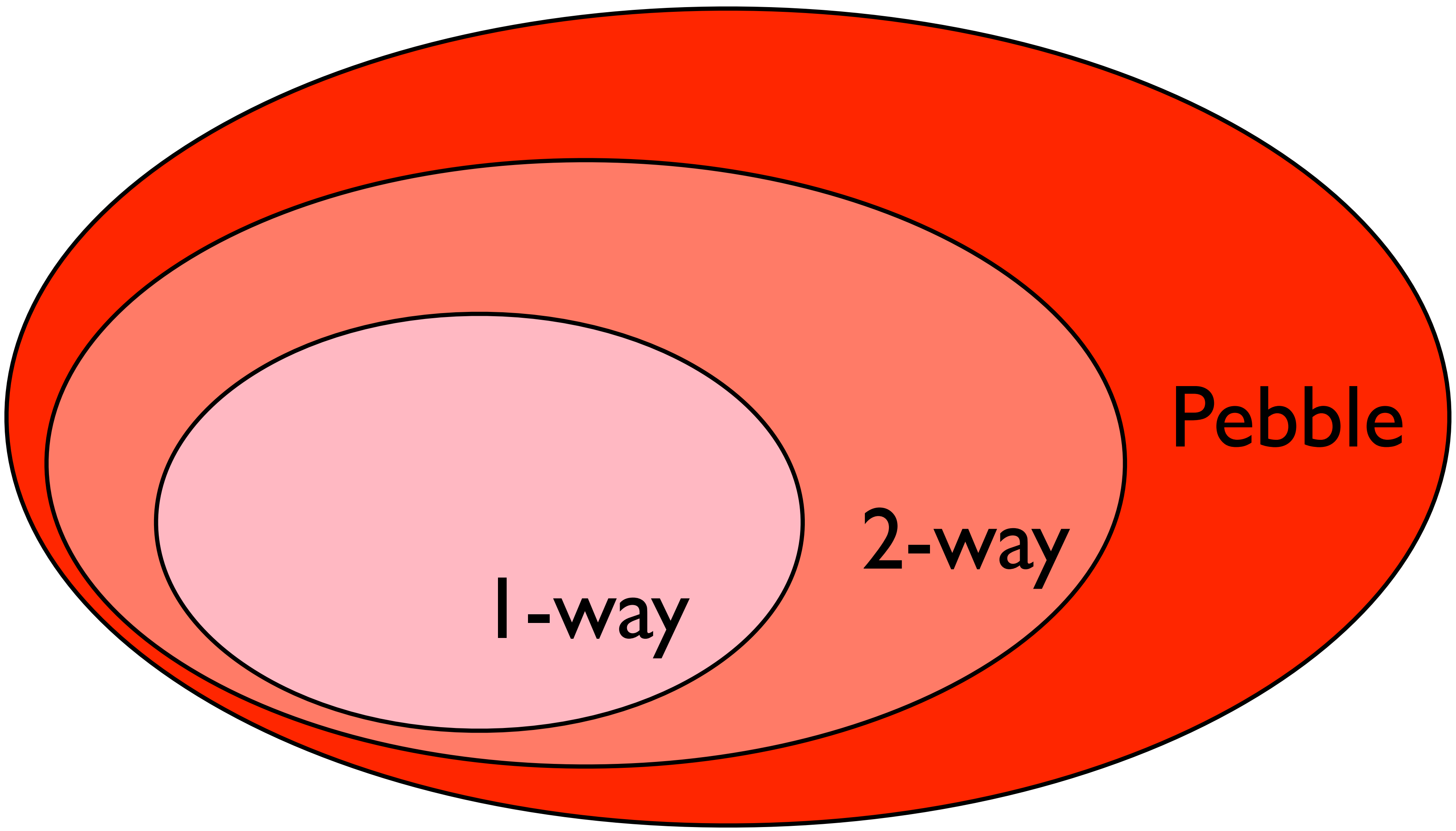
$\mathcal{A}_{\mathsf{F}\,\varphi}(x) =$



$$E_{\mathsf{F}\,\varphi}(x) = \triangleright ? \rightarrow^* x ? \Big((x ! E_{\neg\varphi}(x)) \rightarrow \Big)^* (x ! E_\varphi(x)) \rightarrow^* \triangleleft ?$$

Theorem

- PREs and PAs are expressively equivalent.
- 2-way PREs and 2-way PAs are expressively equivalent.
- Pebble PREs and Pebble PAs are expressively equivalent.



Extensions

- Add 2-way and pebbles in automata and expressions (XPath-like syntax)
- Possibility to *express* more, e.g. *smaller* probabilities (to represent rare events)
- Still a natural way to denote probabilistic properties about words

Conclusion

- General Kleene-Schützenberger theorems for Probabilistic models (classical, two-way, pebbles...)
- Study of Probabilistic Expressions and their extensions permits us to better understand which behavior Probabilistic Automata can generate
- In [1], we proved that Weighted Automata with two-way and pebbles can be evaluated efficiently
- Future work: get logical formalisms generating the same expressivity, and implement quick algorithms to perform translation from PREs to PAs (as there are some for weighted automata, see [2,1] e.g.)

[1] P. Gastin and B. M. (2006). Adding Pebbles to Weighted Automata. In Proceedings of CIAA'12.
[2] C. Allauzen, and M., Mohri, (2006). A Unified Construction of the Glushkov, Follow, and Antimirov Automata. In Proceedings of MFCS'06